Effective electromagnetic geometry

M. Novello* and J. M. Salim
Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, Urca 22290-180 Rio de Janeiro, RJ, Brazil
(Received 25 September 2000; revised manuscript received 27 November 2000; published 22 March 2001)

We show that the propagation of photons in a nonlinear dielectric medium can be described in terms of a modification of the metric structure of space-time. We solve completely the case in which the dielectric constant ϵ is an arbitrary function of the electric field $\epsilon(E)$. The particular case of no dependence on the field reduces to the Gordon metric.

DOI: 10.1103/PhysRevD.63.083511 PACS number(s): 98.80.Bp, 98.80.Cq

I. INTRODUCTION

The electromagnetic force a photon undergoes in a nonlinear regime can be geometrized. This is a rather unexpected result and at the same time a beautiful consequence of the analysis of the behavior of the discontinuities of nonhomogeneous nonlinear electromagnetic field. Original results on this property were presented by Gordon [1]—for the behavior of photons in a rather simple linear dielectric medium in motion—and by Plebansky [2] in the case of Born-Infeld electrodynamics. In the last decade a thorough analysis on the photon propagation in nonlinear electrodynamics was undertaken [3–7]. The net result of all this effort can be summarized (see Ref. [3]) by the statement that "the discontinuities of the electromagnetic field in a nonlinear regime propagate along null geodesics of an effective geometry $g^{\mu\nu}_{(\mathrm{eff})}$ which depends on the energy-momentum distribution of the electromagnetic field." In the case in which the dynamics of the field is described by a Lagrangian L (which depends only on the invariant $F \equiv F^{\mu\nu} F_{\mu\nu}$) the effective metric[1] is given by

$$g^{\mu\nu} = L_F \eta^{\mu\nu} - 4\, L_{FF}\, F^{\mu}{}_{\lambda} F^{\lambda\nu}. \tag{1}$$

in which L_F is the derivative of the Lagrangian L with respect to the invariant F; and similarly for higher order derivatives. The background Minkowski[2] metric tensor is denoted by its standard form $\eta^{\mu\nu}$. Let us point out that this should not be taken as an absolute modification of the geometry of the spacetime, since only the photons paths allow for description in terms of a modification of the metrical properties of the space-time. However, we shall see that in certain situations the dynamical aspects of the field also admit a sort of geometrization. We will make some further comments on this issue in the conclusion.

The general electromagnetic field is defined by two skew-symmetric tensors of rank 2. $F_{\mu\nu}$ is called the electric field-magnetic induction tensor and $P_{\mu\nu}$ is called the electric induction-magnetic field tensor. The dual $F^*_{\mu\nu}$ is defined as

$$F^*_{\alpha\beta} \doteq \frac{1}{2}\, \eta_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}, \tag{2}$$

where $\eta_{\alpha\beta\mu\nu}$ is the completely antisymmetric Levi-Civita tensor.

II. THE METHOD OF THE EFFECTIVE GEOMETRY

We are particularly interested in the derivation of the characteristic surfaces which guide the propagation of the field discontinuities and in the relationship between the properties of the medium and of the associated metric structure. For this purpose we use the Hadamard method in order to obtain the propagation equations for the discontinuities of the electromagnetic field.

With this method we can transpose part of the behavior of photons from the well-known combined Maxwell-Einstein framework to the nonlinear case of electrodynamics. An example of a physical situation where this can be realized will be presented in this paper. It concerns the possibility of the existence of closed paths for photons in spacetime.

Let Σ be a surface of discontinuity for the electromagnetic field. Following Hadamard [8,9] we assume that the field itself is continuous when crossing Σ, while its first derivative presents a finite discontinuity. We accordingly set

$$[F_{\mu\nu}]_\Sigma = 0 \tag{3}$$

and

$$[\partial_\lambda F_{\mu\nu}]_\Sigma = f_{\mu\nu} k_\lambda, \tag{4}$$

in which the symbol

$$[J]_\Sigma \equiv \lim_{\delta \to 0^+} (J|_{\Sigma+\delta} - J|_{\Sigma-\delta})$$

represents the discontinuity of the arbitrary function J through the surface Σ characterized by the equation $\Sigma(x^\mu) = \text{const}$. The tensor $f_{\mu\nu}$ denotes the discontinuity of the field, and

$$k_\lambda = \partial_\lambda \Sigma \tag{5}$$

*Email address: novello@lafex.cbpf.br

[1]From here on we denote the effective metric simply as $g^{\mu\nu}$, once there is no possibility of misunderstanding that this metric concerns only the propagation of the photons.

[2]In this paper we restrict our analysis to the case in which the background metric is flat. Note, however, that all our considerations here can be applied in a curved Riemannian background spacetime.

is the propagation vector. In an analogous way we define

$$[P_{\mu\nu}]_{\Sigma}=0 \tag{6}$$

and

$$[\partial_{\lambda} P_{\mu\nu}]_{\Sigma}=p_{\mu\nu}k_{\lambda}. \tag{7}$$

It is convenient to project these tensors in the framework of a real observer endowed with normalized four-velocity v^{μ}, thus defining the corresponding electric and magnetic vectors in the three-dimensional rest-space of the observer v^{μ}:

$$F_{\mu\nu}=E_{\mu}v_{\nu}-E_{\nu}v_{\mu}+\eta_{\mu\nu}^{\rho\sigma}v_{\rho}B_{\sigma} \tag{8}$$

and

$$P_{\mu\nu}=D_{\mu}v_{\nu}-D_{\nu}v_{\mu}+\eta_{\mu\nu}^{\rho\sigma}v_{\rho}H_{\sigma}. \tag{9}$$

The equations of motion are

$$\partial_{\nu}P^{\mu\nu}=0 \tag{10}$$

and

$$\partial_{\nu}F^{*\mu\nu}=0. \tag{11}$$

In the present article we shall focus our analysis on the case in which the polarization tensor is such that $D_{\alpha}=\epsilon E_{\alpha}$ and $B_{\alpha}=\mu H_{\alpha}$. Besides, we take the dielectric permittivity to be a real function of the electric field, that is $\epsilon=\epsilon(E)$, and the magnetic permittivity μ to be a constant. substituting the definitions (8),(9) in Eqs. (10) and (11), taking in account the special case described above, the field equations becomes

$$\epsilon\partial_{\alpha}E^{\alpha}-\epsilon'\frac{E^{\alpha}E^{\mu}}{E}\partial_{\alpha}E_{\mu}=0, \tag{12}$$

$$\mu\partial_{\mu}H^{\mu}=0, \tag{13}$$

$$\epsilon\dot{E^{\lambda}}-\epsilon' E^{\lambda}\frac{v^{\alpha}E^{\mu}}{E}\partial_{\alpha}E_{\mu}+\eta^{\lambda\beta\rho\sigma}v_{\rho}\partial_{\beta}H_{\sigma}=0, \tag{14}$$

$$\mu\dot{H^{\lambda}}-\eta^{\lambda\beta\rho\sigma}v_{\rho}\partial_{\beta}E_{\sigma}=0, \tag{15}$$

where $\epsilon'=d\epsilon/dE$. Following the definitions and procedure presented above, expressing the discontinuities of the electric and magnetic fields as e^{μ} and h^{μ}, one gets from the discontinuity of Eqs. (12)–(15)

$$\epsilon e^{\mu}k_{\mu}-\frac{\epsilon'}{E}E^{\nu}E^{\mu}e_{\mu}k_{\nu}=0, \tag{16}$$

$$\mu h^{\mu}k_{\mu}=0, \tag{17}$$

$$\epsilon k^{\nu}v_{\nu}e^{\lambda}-\frac{\epsilon'}{E}E^{\nu}e_{\nu}k^{\mu}v_{\nu}E^{\lambda}+\eta^{\lambda\beta\rho\sigma}v_{\rho}h_{\sigma}k_{\beta}=0, \tag{18}$$

$$\mu k^{\nu}v_{\nu}h^{\lambda}-\eta^{\lambda\beta\rho\sigma}v_{\rho}e_{\sigma}k_{\beta}=0. \tag{19}$$

Using Eq. (19) to substitute h_{σ} in Eq. (18) we obtain

$$\epsilon k^{\alpha}v_{\alpha}e^{\lambda}-\frac{\epsilon'}{E}k^{\alpha}v_{\alpha}E^{\beta}e_{\beta}E^{\lambda}+\frac{e^{\lambda}}{\mu k^{\alpha}v_{\alpha}}[k^{\mu}k_{\mu}-(k^{\nu}v_{\nu})^{2}]$$
$$-\frac{k^{\alpha}e_{\alpha}}{\mu k^{\beta}v_{\beta}}k^{\lambda}=0. \tag{20}$$

Multiplying this equation by E_{λ}, using Eq. (16) to eliminate $k^{\alpha}e_{\alpha}$ we get, after some algebraic manipulations

$$\left[\eta^{\mu\nu}+v^{\mu}v^{\nu}(\mu\epsilon-1+\mu\epsilon' E)-\frac{\epsilon'}{\epsilon E}E^{\mu}E^{\nu}\right]k_{\mu}k_{\nu}=0. \tag{21}$$

It then follows that the photon path is kinematically described by

$$g^{\mu\nu}k_{\mu}k_{\nu}=0, \tag{22}$$

where the effective metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu}=\eta^{\mu\nu}+v^{\mu}v^{\nu}(\mu\epsilon-1+\mu\epsilon' E)-\frac{\epsilon' E}{\epsilon}l^{\mu}l^{\nu}, \tag{23}$$

where l^{μ} is the unitary vector in the direction of the electric field. In the particular case in which ϵ is a constant, this formula goes into the reduced Gordon geometry

$$g^{\mu\nu}_{\text{Gordon}}=\eta^{\mu\nu}+v^{\mu}v^{\nu}(\mu\epsilon-1). \tag{24}$$

The inverse metric $g_{\mu\nu}$, defined by $g^{\mu\nu}g_{\nu\alpha}=\delta^{\mu}_{\alpha}$, is given by

$$g_{\mu\nu}=\eta_{\mu\nu}-v_{\mu}v_{\nu}\left(1-\frac{1}{\mu\epsilon(1+\xi)}\right)+\frac{\xi}{1+\xi}l_{\mu}l_{\nu}, \tag{25}$$

where we have set $\xi\equiv\epsilon' E/\epsilon$.

We note that, once the wave vector k_{α} is a gradient, the photon path is a true geodesic in the effective geometry [3]. We obtain then the remarkable result that the discontinuities of the electromagnetic field in a nonlinear electrodynamics propagate along null geodesics of an effective geometry which depends on the properties of the background field.

The velocity of the photon $v_{ph}=k_{\mu}v^{\mu}/|k|$, where $|k|\equiv(\eta^{\mu\nu}-v^{\mu}v^{\nu})k_{\mu}k_{\nu}$, is given by

$$v_{ph}=\frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{1+\xi\cos^{2}\theta}{1+\xi}},$$

in which θ is the angle between the direction of the electric field and the propagation of the photon. Note that in the limit case in which ξ vanishes, the photon velocity coincides with the square root of the determinant of the effective metric. Indeed, for a geometry given by $g_{\mu\nu}=\eta_{\mu\nu}+a_{\mu\nu}$ where $a_{\mu\nu}$ is symmetric, we have

$$\det g_{\mu\nu}=1+a+\frac{1}{2}a^2+\frac{1}{6}a^3+\frac{1}{24}a^4-\frac{1}{2}(\hat{a}\hat{a})+\frac{1}{3}(\hat{a}\hat{a}\hat{a})$$
$$-\frac{1}{4}(\hat{a}\hat{a}\hat{a}\hat{a})-\frac{1}{2}a(\hat{a}\hat{a})+\frac{1}{3}a(\hat{a}\hat{a}\hat{a})-\frac{1}{4}a^2(\hat{a}\hat{a})$$
$$+\frac{1}{8}(\hat{a}\hat{a})^2, \tag{26}$$

where

$$a\equiv a^{\mu}_{\mu}$$
$$\hat{a}\hat{a}\equiv a^{\mu}_{\nu}a^{\nu}_{\mu}$$
$$\hat{a}\hat{a}\hat{a}\equiv a^{\nu}_{\mu}a^{\mu}_{\alpha}a^{\alpha}_{\nu}$$
$$\hat{a}\hat{a}\hat{a}\hat{a}\equiv a^{\nu}_{\mu}a^{\mu}_{\alpha}a^{\alpha}_{\beta}a^{\beta}_{\nu}. \tag{27}$$

Using this property it follows that the determinant of the effective metric is given by

$$\det g_{\mu\nu}=\frac{1}{\mu\epsilon(1+\xi)^2}. \tag{28}$$

In the case where ϵ does not depend on the electric field, the photon velocity can be written in terms of the determinant of the effective metric

$$v_{ph}=\sqrt{g}. \tag{29}$$

We would like to remark that Eqs. (28) and (29) are not tensorial equations. They are valid only when the Minkowski frame is expressed in Cartesian coordinates.

An inspection on the restricted effective metric of the particular case examined by Gordon allows us to make the following statements [10,11]:

The constitutive relations can be displayed in terms of the restricted Gordon geometry.

Indeed, if we set

$$P^{\mu\nu}=\frac{1}{\mu}\tilde{F}^{\mu\nu},$$

where

$$\tilde{F}^{\mu\nu}\equiv g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}.$$

Then, $P_{\alpha\beta}v^{\beta}=D_{\alpha}=\epsilon E_{\alpha}$ and correspondingly, $B_{\alpha}=\mu H_{\alpha}$.

The dynamics of the electromagnetic field in a linear medium can be written in terms of the Gordon optical metric.

This is nothing but a consequence of the following fact. The dynamical equations of the electromagnetic field in a medium (ϵ,μ) are

$$F_{\{\mu\nu,\alpha\}}=0$$

and

$$P^{\alpha\beta}{}_{,\beta}=0.$$

This last equation, written in terms of the tensor $F_{\mu\nu}$, yields

$$\left(\sqrt{\frac{\epsilon}{\mu}}\tilde{F}^{\alpha\beta}\right)_{;\beta}=0,$$

where the semicolon stands for the covariant derivative in the Gordon geometry.

These properties are restricted to the particular case of the Gordon metric and are not valid anymore in a general nonlinear medium in which the dielectric function ϵ depends on the electric field. Thus, the fact that the Gordon metric has a double role—that is, it permits us to rewrite the dynamical equations of the electromagnetic field in a polarized medium and guides the evolution of the discontinuities of the field—is just a miracle for linear media and cannot be generalized for the nonlinear structure. Note, however, that such a generalization procedure is possible to be undertaken for certain particular dynamics driven by nonlinear Lagrangians. A remarkable example of this case is Born-Infeld electrodynamics. We will analyze this in a forthcoming paper.

III. FINAL COMMENTS

In this paper we have generalized Gordon geometry that describes the behavior of photons in a moving dielectric medium. Our result is valid for the case of an arbitrary dependence of the dielectric permittivity ϵ on the electric field $\epsilon=\epsilon(E)$. This allows us to interpret such result as being nothing but the proof that the electromagnetic force that acts on the photon can be geometrized. In the linear case such geometrization can be further extended for the dynamics of the electromagnetic field itself. In the case that ϵ is a function of the electric field this generalization is not possible.

ACKNOWLEDGMENTS

The authors are grateful to Dr. S. E. Perez Bergliaffa and Dra. M. C. Motta da Silva for reading the manuscript.

[1] W. Gordon, Ann. Phys. (Leipzig) **72**, 421 (1923).
[2] J. Plebansky, *Lectures on Non-linear Electrodynamics* (Nordita, Copenhagen, 1968).
[3] M. Novello, V. A. De Lorenci, J. M. Salim, and R. Klippert, Phys. Rev. D **61**, 045001 (2000).
[4] M. Novello, V. De Lorenci, and E. Elbaz, Int. J. Mod. Phys. A **13**, 4539 (1998).
[5] W. Dittrich and H. Gies, Phys. Rev. D **58**, 025004 (1998).
[6] V. A. De Lorenci, R. Klippert, M. Novello, and J. M. Salim, Phys. Lett. B **482**, 134 (2000).

[7] Ulf Leonhardt, Phys. Rev. A **62**, 012111 (2000).
[8] J. Hadamard, *Leçons sur la Propagation des Ondes et les Equations de l' Hydrodynamique* (Dunod, Paris, 1958).
[9] Y. Choquet-Bruhat, C. De Witt-Morette, and M. Dillard-Bleick, in *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1977).
[10] J. Ehlers, in *Perspectives in Geometry and Relativity (Essays in Honor of V. Hlavaty)* (Indiana University Press, Bloomington, IN, 1966).
[11] V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Vol. 99 of International Series in Natural Philosophy (Pergamon, New York, 1979).