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ON A GEOMETRICAL DESCRIPTION OF QUANTUM MECHANICS

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We show that quantum mechanics can be interpreted as a modification of the Euclidean nature of 3-d space into a particular affine space, which we call Q-wis. This is proved using the Bohm–de Broglie causal formulation of quantum mechanics. In the Q-wis geometry, the length of extended objects changes from point to point. In this formulation, deformation of physical distances are in the core of quantum effects allowing a geometrical formulation of the uncertainty principle.

Keywords: Foundations of quantum mechanics; Bohm–de Broglie interpretation; Weyl integrable space; non-Euclidean geometry.

1. Introduction

The early years of quantum mechanics were marked by intense debates and controversies related to the meaning of the new behavior of matter. While one group was convinced that it was unavoidable to abandon the classical picture, the other group tried incessantly to save its main roots and conceptual pillars. To be able to reproduce the atypical quantum effects, the latter group was forced to introduce new ingredients such as de Broglie's pilot wave [1–6] or Mandelung's hydrodynamical picture [7].

However, the lack of physical explanations for these *ad hoc* modifications weakened these pictures. At the same time, the former group leaded by Schrödinger, Bohr and Heisenberg was increasingly gaining new adepts until its climax in the 1927 Solvay's conference when this picture was finally accepted as the orthodox interpretation of quantum mechanics — the Copenhagen interpretation [8–10].

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Notwithstanding, a marginal group of physicists continued to develop other approaches [11–13] to describe quantum mechanics that are more adequate to connect to a classical picture.[a] One of the most prominent amongst these alternative interpretations is the causal interpretation of quantum mechanics also known as Bohm–de Broglie interpretation [14–17].

The development of quantum cosmological scenario brought to light some difficulties intrinsic to the Copenhagen interpretation. More specifically, the measurement process in a quantum closed universe seems inevitably inconsistent [18–20]. Fortunately, there are some alternative interpretations that are consistently applied simultaneously to cosmology and to the micro-world. As two examples we mention the many-worlds interpretation [18, 21, 22] and the consistent histories formulation [23–26].

In the present work we will focus only on the Bohm–de Broglie interpretation since it is amongst the well-defined interpretation that can be applied to any kind of system, and up to date it is completely equivalent to the Copenhagen interpretation when applied to the micro-world.

We will show that it is possible to interpret all quantum phenomena as a modification of the geometrical properties of the physical space. Hence, we will deal with a generalization of the Euclidean geometry. Several papers in the literature have advocated a possible connection between non-Euclidean geometry and quantum effects [29–43]. Despite the different approaches, a common feature of all these works is the proposal of a new geometry that was first introduced by Weyl [27, 28]. The so-called Weyl geometry is a modification of a Riemannian space-time to accommodate the conformal map as a pure gauge transformation. As it is well known, this can be achieved only if not just the metric but all fields are also conformally transformed. In addition, one has to redefine the covariant derivative to maintain the Riemannian structure of the space-time.

Instead of working with this Weyl geometry, we shall define a different space where the connection cannot be specified solely in terms of the metric and which we shall call Q-wis.[b] There are at least two main differences. The Q-wis is not conformally invariant nor is there any kind of geometrical gauge degree of freedom. Furthermore, the covariant derivative is strictly defined only with the connection so that the covariant derivative of the metric does not vanish. Contrarily to the Weyl geometry, the affine degree of freedom of the Q-wis space raises physical implication that allows us to re-interpret the quantum effects. In fact, we shall propose a physical description of quantum effects through a limitation of the classical standards,

[a]Since we are not concerned with relativistic phenomena, the term classical physics should be understood as pre-relativistic physics unless otherwise specified.

[b]This name is an acronym for quantum Weyl integrable space that is motivated from the fact that it is similar to a Weyl integrable space and at the same time describes quantum phenomena. Its mathematical properties are analyzed in the Appendix.

or in other words, through a limitation of the Euclidean standards used to measure physical distances.

The Appendix is reserved to develop in more detail the properties of a Q-wis space, but it is worth to mention its main difference from an Euclidean space that is related to the notion of a standard ruler.

A Q-wis is a geometrical space endowed with an Euclidian metric. Furthermore, this space also posses an extra degree of freedom, which allows the length of a vector to change from point to point. This means that a ruler of length l if parallel transported will change by an amount

$$\delta l = l f_{,a} \mathrm{d}x^a. \tag{1.1}$$

A Q-wis is distinguished precisely by the fact that the length of the ruler transported along a closed curve does not change. Hence, if the change of the ruler's length is $\mathrm{d}l$, for a closed path in Q-wis we have

$$\oint \mathrm{d}l = 0, \tag{1.2}$$

which guarantees the uniqueness of any local measurement. The allowance of an intrinsic modification of the standard rulers is the main geometrical hypothesis of the present work. We shall argue how this geometrical modification can be in the origin of quantum effects. For the sake of clarity we will deal with the simplest system possible, namely an isolated point-like particle possibly subjected to an external potential.

The outline of the paper is as follows. In the next section, we briefly review the main points of quantum mechanics and in the section *Non-Euclidean geometry*, we describe how to connect the Q-wis space to the quantum theory. We show that quantum mechanics can be derived from a geometrical variational principle. Then, in the conclusions we present our final remarks. The Appendix is reserved to describe the main properties of the Q-wis geometry.

2. Quantum Mechanics

Quantum mechanics is a modification of the classical laws of physics to incorporate the uncontrolled disturbance caused by the macroscopic apparatus necessary to realize any kind of measurement. This statement, known as Bohr's complementary principle, contains the main idea of the Copenhagen interpretation of quantum mechanics. The quantization program continues with the correspondence principle promoting the classical variables into operators and the Poisson brackets into commutation relations.

In this non-relativistic scenario, the Schrödinger equation establishes the dynamics for the wave function describing the system. Note that as in Newtonian mechanics, time is only an external parameter and the 3-d space is assumed to be endowed with the Euclidean geometry.

Using the polar form for the wave function, $\Psi = Ae^{iS/\hbar}$, the Schrödinger equation can be decomposed in two equations for the real functions $A(x)$ and $S(x)$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} = 0, \tag{2.1}$$

$$\frac{\partial A^2}{\partial t} + \nabla\left(A^2\frac{\nabla S}{m}\right) = 0. \tag{2.2}$$

Solving these two equations is completely analogous to solving the Schrödinger equation. The probabilistic interpretation of quantum mechanics associate $\|\Psi\|^2 = A^2$ with the probability distribution function on configuration space. Hence, Eq. (2.2) has exactly the form of a continuity equation with $A^2\nabla S/m$ playing the role of current density.

2.1. *Bohm–de Broglie interpretation*

The causal interpretation, which is an ontological hidden variable formulation of quantum mechanics, propose that the wave function does not contain all the information about the system.

An isolated system describing a free particle (or a particle subjected to a potential V) is defined simultaneously by a wave function and a point-like particle. In this case, the wave function still satisfies the Schrödinger equation but it should also work as a guiding wave modifying the particle's trajectory.

Note that Eq. (2.1) is a Hamilton–Jacobi-like equation with an extra term that is often called quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}, \tag{2.3}$$

while, as already mentioned, Eq. (2.2) is a continuity-like equation. The Bohm–de Broglie interpretation takes these analogies seriously and postulate an extra equation associating the velocity of the point-like particle with the gradient of the phase of the wave function. Hence,

$$\dot{x} = \frac{1}{m}\nabla S. \tag{2.4}$$

Integrating Eq. (2.4) yields the quantum Bohmian trajectories. The unknown or hidden variables are the initial positions necessary to fix the constant of integration of the above equation.

For the case of a spinless particle, the quantum potential is solely responsible for all novelties of quantum effects such as non-locality or tunneling processes. As a matter of fact, the Bohm–de Broglie interpretation has the theoretical advantage of having a well formulated classical limit. Classical behavior is obtained as soon as the quantum potential, which has dimensions of energy, becomes negligible compared to other energy scales of the system.

In what follows, we will show that it is possible to reinterpret quantum mechanics as a manifestation of non-Euclidean structure of the 3-d space. Hence, we propose a geometrical interpretation to describe quantum effects.

3. Non-Euclidean Geometry

Since ancient times, Euclidean geometry was considered as the most adequate mathematical formulation to describe the physical space. However, its validity can only be established *a posteriori* as long as its construction yields useful notions to connect physical quantities such as the Euclidean distance between two given points.

Special relativity modified the notion of three-dimensional Euclidean space to incorporate time in a four-dimensional continuum (Minkowski space-time). Later on, General Relativity generalized the absolute Minkowski space-time to describe gravitational phenomena. General Relativity considers the spacetime manifold as a dynamical field that can be deformed and stretched but in such a way that it always preserves its Riemannian structure. It is worth noting that both the Euclidean and Minkowskian spaces are nothing more than special cases of Riemannian spaces.

Nonetheless, Riemannian manifold are not the most general type of geometrical spaces. In the same way as above, Riemannian geometries can be understood as a special subclass of a more general structure where the connection is not uniquely determined by the metric. Geometries where the connection is not just the Christoffel's symbol are known as affine space. As to the matter of which geometry is actually realized in Nature, it has to be determined by physical experiments.

Instead of imposing *a priori* that quantum mechanics has to be constructed over an Euclidean background as it is traditionally done, we shall argue that quantum effects can be interpreted as a manifestation of a non-Euclidean structure derived from a variational principle. The validity of the specific geometrical structure proposed can be checked *a posteriori* comparing it to the usual non-relativistic quantum mechanics.

Thus, consider a point-like particle with velocity $v = \frac{\nabla S}{m}$ and subjected to a potential V. We shall follow Einstein's idea to derive the geometrical structure of space from a variational principle by considering the connection as an independent variable and hence by using Palatini's variational procedure. The validity of an action principle can only be justified *a posteriori* by deriving the correct dynamical equation of motion but normally its formulation already specifies the kinematical properties of the theory. In particular, we consider an action that includes geometry and the particle's Lagrangian with the peculiarity that the particle is non-minimally coupled to geometry through a scalar field Ω which we shall show to be related to the affine structure of 3-d space. Thus we define the action by

$$I = \int \mathrm{d}t\, \mathrm{d}^3x \sqrt{g}[\lambda^2 \Omega^2 \mathcal{R} - \Omega^2 \mathcal{L}_m], \tag{3.1}$$

where the connection of the 3-d space Γ^i_{jk}, the Hamilton's principal function S and the scalar function Ω should be understood as independent variables. Each term in Eq. (3.1) is defined as follows: we are considering the line element in Cartesian coordinates given by

$$ds^2 = g_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2 \tag{3.2}$$

with

$$g = \det g_{ij}. \tag{3.3}$$

The Ricci curvature tensor is defined in term of the connection through

$$\mathcal{R}_{ij} = \Gamma^m_{mi,j} - \Gamma^m_{ij,m} + \Gamma^l_{mi}\Gamma^m_{jl} - \Gamma^l_{ij}\Gamma^m_{lm} \tag{3.4}$$

and its trace defines the curvature scalar $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij}$ which has dimensions of inverse length squared, $[\mathcal{R}] = L^{-2}$. The constant λ^2 has dimension of energy times length squared, $[\lambda^2] = E \cdot L^2$, and the particle's Lagrangian is defined by the Hamilton's function through

$$\mathcal{L}_m = \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V, \tag{3.5}$$

where $\frac{\partial S}{\partial t}$ is related to the particle's total energy.

From Eq. (3.1), variation of the action I with respect to the independent variables gives respectively (see Appendix for details)

$$\delta\Gamma^i_{jk} : g_{ij;k} = -4(\ln\Omega)_{,k}g_{ij}, \tag{3.6}$$

where ";" denotes covariant derivative and a common "," simple spatial derivative. Equation (3.6) characterizes the affine properties of the physical space. Hence, the variational principle naturally defines a Q-wis space. Variation with respect to Ω gives

$$\delta\Omega : \lambda^2\mathcal{R} = \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V. \tag{3.7}$$

The right-hand side of this equation has dimension of energy while the curvature scalar has dimension of $[\mathcal{R}] = L^{-2}$. Furthermore, apart from the particle's energy, the only extra parameter of the system is the particle's mass m. Thus, there is only one-way to combine the unknown constant λ^2, which has dimension of $[\lambda^2] = E \cdot L^2$, with the particle's mass such as to form a physical quantity. Multiplying them, we find a quantity that has dimension of angular momentum squared $[m \cdot \lambda^2] = \hbar^2$.

In terms of the scalar function Ω, the curvature scalar is given by (see Appendix)

$$\mathcal{R} = 8\frac{\nabla^2\Omega}{\Omega}. \tag{3.8}$$

Hence, setting $\lambda^2 = \frac{\hbar^2}{16m}$, Eq. (3.7) becomes

$$\delta\Omega : \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega} = 0. \tag{3.9}$$

Finally, varying the Hamilton's principal function S we find

$$\delta S : \frac{\partial \Omega^2}{\partial t} + \nabla \left(\Omega^2 \frac{\nabla S}{m} \right) = 0. \tag{3.10}$$

Equations (3.9) and (3.10) are identical to Eqs. (2.1) and (2.2) if we identify $\Omega = A$. Thus, the "action" of a point-like particle non-minimally coupled to geometry is given by

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\Omega^2 \left[\frac{\hbar^2}{16m}\mathcal{R} - \left(\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S \cdot \nabla S + V \right) \right]$$

exactly reproduce the Schrödinger equation and thus the quantum behavior. The straightest way to compare this geometrical approach to the common quantum theories is to relate it to the Bohm–de Broglie interpretation.[c] Note that this formulation has the advantage of giving a physical explanation of the appearance of the quantum potential, Eq. (2.3). In a Q-wis, this term is simply its curvature scalar. The inverse square root of the curvature scalar defines a typical length L_w (Weyl length) that can be used to evaluate the strength of quantum effects

$$L_w \equiv \frac{1}{\sqrt{\mathcal{R}_w}}. \tag{3.11}$$

As we have already mentioned, the classical limit of Bohm–de Broglie interpretation is achieved when the quantum potential is negligible compared to other energy scales of the system. In the scope of this geometrical approach, the classical behavior is recovered when the length defined by the Q-wis curvature scalar is small compared to the typical length scale of the system. Once the Q-wis curvature becomes non-negligible the system goes into a quantum regime.

3.1. *Geometrical uncertainty principle*

As long as we accept that quantum mechanics is a manifestation of a non-Euclidean geometry, we are faced with the need of reinterpreting geometrically all theoretical issues related to quantum effects. As a first step, we associate the uncertainty principle to a break down of the classical notion of a standard ruler.

In Euclidian space, there is a clear notion of distance between two points. Generalizing to curved spaces, it is still possible to define distance as the smallest length between two given points calculated along geodesics in 3-d space. This is a consistent definition since the 3-d space has a true metric in the mathematical sense that its eigenvalues are all positives. However, this definition does not encompass the classical notion of a standard ruler since standard rulers are based on Euclidean space. This means that if quantum mechanics can be interpreted as a modification

[c] Up to date, all interpretation of quantum mechanics are on equal footing. Thus, establishing the connection with the causal interpretation automatically links this geometrical interpretation with all others.

of Euclidean space it shall have a limit of validity for the notion of Euclidean distance. In other words, it should not be possible to perform a classical measurement of distances smaller than a given value that of course should be related to the curvature scale of the space, i.e. the Q-wis curvature length. Thus, we propose that any measurement can only measure distances bigger than the Weyl length

$$\Delta L \geq L_w = \frac{1}{\sqrt{\mathcal{R}_w}}. \tag{3.12}$$

The quantum regime is extreme when the Q-wis curvature term dominates. Thus, from Eqs. (3.8) and (3.9) we have

$$\mathcal{R}_w = 2\left(\frac{2\Delta p}{\hbar}\right)^2 - \frac{16m}{\hbar^2}(E - V) \leq 2\left(\frac{2\Delta p}{\hbar}\right)^2 \tag{3.13}$$

and finally combining Eqs. (3.12) and (3.13) we obtain

$$\Delta L \cdot \Delta p \geq \frac{\hbar}{2\sqrt{2}}. \tag{3.14}$$

We should emphasize that now the Heisenberg's uncertainty relation has a pure geometrical meaning. Our argument closely resembles Bohr's complementary principle inasmuch as the impossibility of applying the classical definitions of measurements. However, we strongly diverge with respect to the fundamental origin of the physical limitation.

Bohr's complementary principle is based on the uncontrolled interference of a classical apparatus of measurement. On the other hand, we argue that the notion of a classical standard ruler breaks down because its meaning is intrinsically dependent on the validity of Euclidean geometry. Once it becomes necessary to include the Q-wis curvature, we are no longer able to perform a classical measurement of distance.

There is another way to interpret the uncertainty principle. First, recall that in scattering processes one can define the classical electron radius as $r_e \equiv \frac{e^2}{mc^2}$ since it has dimension of length and it is the classical radius for which the electrostatic self-energy is equal to the electron mass. Furthermore, the Thomson cross section is approximately the area defined with this classical radius, $\sigma_T \approx 4\pi r_e^2$.

Now, we shall construct a similar notion for our quantum system. For a given particle of mass m and energy E there is only one combination with the free parameter of the theory λ which has dimension of length, i.e. $\frac{\lambda}{\sqrt{E}}$. We take this value as a definition of the classical size of the particle, namely

$$l_{\text{part}} \equiv \frac{\lambda}{\sqrt{E}} = \sqrt{\frac{\hbar^2}{16mE}}. \tag{3.15}$$

One might worry of the appearance of a $\hbar$ in the definition of a classical quantity. However, as we shall show below, this length also establishes how far the system is from a quantum regime which naturally should depend on $\hbar$. The appearance of the Planck constant in our definition is completely analogous to the appearance of the speed of light c for the classical electron radius in non-relativistic Thomson cross-section. The above classical size l_{part} has the same meaning to quantum processes as the classical electron radius has for the Thomson scattering.

Note that this definition coincides with the particle's Compton wavelength if one uses the relativistic relation $E = mc^2$ which is its rest mass potential energy. For a non-relativistic particle the contribution to its total energy comes mainly from its rest mass. Thus, even though the particle might have a kinetic energy, for a non-relativistic particle one is still allowed to use the above equation and compare it to the particle's Compton wavelength inasmuch the Compton wavelength specifies the limits of validity of non-relativistic quantum mechanics. As soon as the system attains the relativistic regime, not only the above equation is no longer valid but also non-relativistic quantum mechanics. Hence, the above equation has the same range of validity as non-relativistic quantum mechanics.

In connection with the definition of a classical radius from the Thomson cross-section we shall conceive a free stationary particle. Moreover, the radius l_{part} defines a volume which we suppose to be at rest so that its energy is related to the curvature through

$$E = \frac{\hbar^2}{16m}\mathcal{R}_W \Rightarrow l_{\mathrm{part}} = \frac{1}{\sqrt{\mathcal{R}_W}}. \tag{3.16}$$

Notwithstanding this finite-size picture, the system describes a point-like particle. Thus, from Eq. (3.13) we can relate the volume defined by l_{part} with the particle's momentum through

$$l_{\mathrm{part}} \cdot \Delta p \geq \frac{\hbar}{2\sqrt{2}}. \tag{3.17}$$

From this point of view, the uncertainty principle indicates that it is impossible to perform a measurement smaller than the classical size of the particle defined by Eq. (3.15). In other words, it is impossible to perform a classical measurement inside what one would normally call a classical particle. This geometrical uncertainty relation attribute to a point-like particle an effective size due to the Q-wis curvature of the 3-d space.

4. Conclusions

It is well known that as soon as we consider high velocities or high energies, one has to abandon the Euclidean geometry as a good description of the physical space. These brought two completely different modifications where the physical space loses its absolute and universal character. In fact, this is the core of classical relativistic physical theories, namely Special and General Relativity.

In a similar way, one should be allowed to consider that the difficulties that appear while going from classical to quantum mechanics come from an inappropriate extrapolation of the Euclidean geometry to the micro-world. Hence, the unquestioned hypothesis of the validity of the 3-d Euclidean geometry to all length scales might be intrinsically related to quantum effects.

In the present work, we have shown that there is a close connection between the Bohm–de Broglie interpretation of quantum mechanics and the Q-wis spaces. In fact, we point out that the Bohmian quantum potential can be identified with the curvature scalar of the Q-wis. Moreover, we present a variational principle that reproduces the Bohmian dynamical equations considered up to date as equivalent to Schrödinger's quantum mechanics.

The Palatini-like procedure, in which the connection acts as an independent variable while varying the action, naturally endows the space with the appropriate Q-wis structure. Thus, the Q-wis geometry enters into the theory less arbitrarily than the implicit *ad hoc* Euclidean hypothesis of quantum mechanics.

The identification of the Q-wis curvature scalar as the ultimate origin of quantum effects leads to a geometrical version of the uncertainty principle. This geometrical description considers the uncertainty principle as a breakdown of the classical notion of standard rulers. Thus, it arises an identification of quantum effects to the length variation of the standard rulers.

Appendix A. Q-Wis Geometry

In this section we shall briefly review the mathematical properties of such 3-d Q-wis space. Contrary to the Riemannian geometry, which is completely specified by a metric tensor, the Q-wis space defines an affine geometry. This means that the covariant derivative which is defined in terms of a connection Γ^m_{ik} depends not only on the metric coefficients but also on the gradient of a scalar function $f_{,a}(x)$. For instance, given a vector X_a its covariant derivative is

$$X_{a;b} = X_{a,b} - \Gamma^m_{ab} X_m. \tag{A.1}$$

The non-metricity of the Q-wis geometry implies that rulers, which are standards of length measurement, changes while we transport it by a small displacement $\mathrm{d}x^i$. This means that a ruler of length l will change by

$$\delta l = l f_{,a} \mathrm{d}x^a. \tag{A.2}$$

Note that even though it changes from point-to-point in a Q-wis, the length does not change along a closed path

$$\oint \mathrm{d}l = 0. \tag{A.3}$$

Contrary to a Riemannian space, the covariant derivative of the metric does not vanish but it is given by

$$g_{ab;k} = f_{,k} g_{ab}. \tag{A.4}$$

Using Cartesian coordinates, it follows that the expression for the connection in terms of $f_{,k}$ takes the form

$$\Gamma^k_{ab} = -\frac{1}{2}(\delta^k_a f_{,b} + \delta^k_b f_{,a} - g_{ab} f^{,k}). \tag{A.5}$$

As a matter of convenience, we define $f = -4\ln\Omega$. The Ricci tensor equation (3.4) constructed with the above connection Eq. (A.5) is given by

$$\mathcal{R}_{ij} = 2\frac{\Omega_{,ij}}{\Omega} - 6\frac{\Omega_{,i}\Omega_{,j}}{\Omega^2} + 2g_{ij}\left[\frac{\nabla^2\Omega}{\Omega} + \frac{\vec{\nabla}\Omega\cdot\vec{\nabla}\Omega}{\Omega^2}\right] \tag{A.6}$$

and its trace the scalar of curvature $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij} = 8\frac{\nabla^2\Omega}{\Omega}$. In the present paper we have used a variational principle, which proof is as follows. Consider the action

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\Omega^2\mathcal{R} \tag{A.7}$$

then, variation of the connection yields

$$\delta I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\Omega^2 g^{ab}\delta R_{ab} = \int \mathrm{d}t\mathrm{d}^3x Z^{ab}_m\delta\Gamma^m_{ab} \tag{A.8}$$

with

$$Z^{ab}_m \equiv (\sqrt{g}g^{ab}\Omega^2)_{;m} - \frac{1}{2}(\sqrt{g}g^{ak}\Omega^2)_{;k}\delta^b_m \pm \frac{1}{2}(\sqrt{g}g^{bk}\Omega^2)_{;k}\delta^a_m. \tag{A.9}$$

Taking its trace we obtain $(\sqrt{g}g^{ak}\Omega^2)_{;k} = 0$. Substituting this expression in (A.9) we finally obtain the condition for a Q-wis geometry

$$g_{ab;k} = -4\frac{\Omega_{,k}}{\Omega}g_{ab}. \tag{A.10}$$

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