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Gaussian Coordinate Systems for the Kerr Metric

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Abstract—We present a class of Gaussian coordinate systems for the Kerr metric obtained from the relativistic Hamilton-Jacobi equation. We discuss the Cauchy problem of such a coordinate system. In the appendix, we present the JEK (Jordan-Ehlers-Kundt) formulation of General Relativity—the so-called quasi-Maxwell equations—which acquires a simpler form in the Gaussian coordinate system. We show how this set of equations can be used to regain the internal metric of the Schwarzschild solution and, with this in mind, we suggest a possible way to find out a physically significant internal solution for the Kerr metric.

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1. INTRODUCTION

The recognition that natural processes cannot be influenced by any choice of representation of the events occurring in space-time leads the covariance principle to be assumed as one of the fundamentals of modern physics. This idea was explicitly used by Einstein to build a theory of gravity, General Relativity, which took a step forward by invoking that the MMG (Manifold Mapping Group) should be taken as an invariance principle of the theory. In practical uses, however, one is always obliged to select a particular language by choosing a special coordinate system to describe a given phenomenon. Among all the possible choices one can make—most of them dictated by symmetries of a given problem—there is a very special one that can bring us more physical insight about the problem to be treated, that is the *Gaussian coordinate system* or *Synchronic coordinate system* (cf. [1]). In fact, this coordinate system was suggested by C. Gauss in his works about curves and surfaces.

Nowadays, the most frequently used solutions of the Einstein equations are expressed, for instance, the Friedmann and Kasner solutions [2, 3], or can be easily expressed in Gaussian coordinate systems (see *the Novikov coordinates* for the Schwarzschild case [4]). The Russian school is a considerable example of applicability of the Gaussian coordinates in general relativity, as we can see in many seminal papers by Lifshitz, Belinski, Khalatnikov, Novikov and others [2, 4, 5, 6].

In a Gaussian coordinate system (GCS), a foliation of space-time is made in such a way that one separates space and time, as in pre-relativistic theories. The timelike world line of an observer that is orthogonal to the 3D space (which is identified with the comoving system) is such that the proper time of that observer coincides with the coordinate time.

In the standard procedure made by the founders of relativistic cosmology, the Gaussian coordinate system seems closely related to the Cosmological Principle: the large-scale structure of the Universe behaves as being homogeneous and isotropic. Solutions of the Einstein equations which satisfy this postulate possess a complete Gaussian coordinate system; in addition, the Gaussian surface is a Cauchy surface. On the other hand, there are cosmological solutions which do not satisfy this postulate. However, these solutions have undesirable properties, for instance, closed timelike curves (CTC's).

A similar approach was realized for the Gödel metric by one of us in [7]. The Gaussian coordinate system was shown to be limited and unable to be extended beyond a certain region, the domain of which depends only on the vorticity present in this geometry. Such an inaccessible region determines a frontier for timelike geodesics, which prohibits extension of the GCS throughout the whole manifold.

Mathematically, a Gaussian coordinate system is constructed by determining a hypersurface $S = S(x^\mu)$ which satisfies

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 1, \quad g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial \bar{x}^i}{\partial x^\nu} = 0, \qquad (1)$$

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where $\bar{x}^i$ are coordinates lying on S. The Gaussian coordinates are given by $\bar{x}^\mu = (S, \bar{x}^i)$. As a natural consequence, Eq. (1) imposes $\bar{g}^{00} = 1$ and $\bar{g}^{0i} = 0$.

The main purpose of this paper is to exhibit a Gaussian coordinate system for the Kerr metric. The method we use is provided by the relativistic Hamilton-Jacobi formalism of canonical transformations, therefore, it cannot be applied to null-like geodesics. The key idea is to identify the principal Hamilton function with the proper time of a test particle in this geometry. An immediate application of this method can be used for other solutions of Einstein's equation, as the Schwarzschild and Kerr-Newman solutions. Finally, in Appendix A we list the coordinate systems encountered in the literature for the Kerr metric including our Gaussian coordinate system, and in Appendix B, from the *Quasi-Maxwell formalism* (JEK equations together with the evolution equations of the kinematical quantities) we regain the Schwarzschild solution in a Gaussian coordinate system.

2. HAMILTON-JACOBI FORMALISM AND GAUSSIAN COORDINATES

In this section, we briefly present the well-known Hamilton-Jacobi (HJ) formalism of canonical transformations. There are several books (cf. at least [1, 8] or [9]) addressing this issue in the literature. Notwithstanding, according to the authors' knowledge, GCS deduction from the HJ has not yet been seen. Then, we shall show this simple calculation here.

The action for a free test particle with mass $m \neq 0$ in the presence of a gravitational field can be written as

$$S = -m \int ds. \tag{2}$$

From the variational principle we discover how S depends functionally on the coordinates x^μ,

$$\delta S = -m u_\alpha \delta x^\alpha, \tag{3}$$

where u_α is the 4-velocity. The 4-momentum p_α defined as

$$p_\alpha \doteq -\frac{\partial S}{\partial x^\alpha}, \tag{4}$$

satisfies

$$p_\alpha p^\alpha = m^2. \tag{5}$$

Substituting Eq. (4) in (5), we find the relativistic Hamilton-Jacobi equation for a test particle in a gravitational field given by

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} - m^2 = 0. \tag{6}$$

Now, by a canonical transformation of coordinates, so that (q,p) goes to (Q,P) via a generating function $F_1(q,Q,t)$, we have the following constraint:

$$dF_1 = \sum_i (p_i dq_i - P_i dQ_i) + (K - H)dt. \tag{7}$$

In a covariant form, we write this exact differential as

$$dF_1(q^\mu, Q^\mu) = -p_\alpha dq^\alpha + P_\mu dQ^\mu, \tag{8}$$

where $p_\alpha = (H, -\vec{p})$ and $P_\mu = (K\partial t/\partial Q^0, -\vec{P})$. Using another generating function $F_2(q^\mu, P_\nu)$, such that it is associated to F_1 by

$$F_2 = F_1 - P_\mu Q^\mu, \tag{9}$$

we have a differential relation for F_2:

$$dF_2 = -p_\alpha dq^\alpha - Q^\mu dP_\mu. \tag{10}$$

Thus the function F_2 must satisfy

$$p_\alpha = -\frac{\partial F_2}{\partial q^\alpha}, \quad Q^\mu = -\frac{\partial F_2}{\partial P_\mu}. \tag{11}$$

Comparing this equation with Eq. (4), we identify the generating function $F_2(q,P)$ as being the action $S(q,P) = S(q, p(q,P))$ of a test particle in a gravitational field. Therefore,

$$p_\alpha = -\frac{\partial S}{\partial q^\alpha}, \tag{12a}$$

$$Q^\mu = -\frac{\partial S}{\partial P_\mu}. \tag{12b}$$

Assuming that the "new" coordinate Q^0 is $Q^0 = -S/m$, from the 0-component of Eq. (12b) we get

$$\frac{S}{m} = \frac{\partial S}{\partial P_0}, \tag{13}$$

therefore, a solution of this equation can be expressed as

$$S = e^{P_0/m} f(q^\mu, P_i). \tag{14}$$

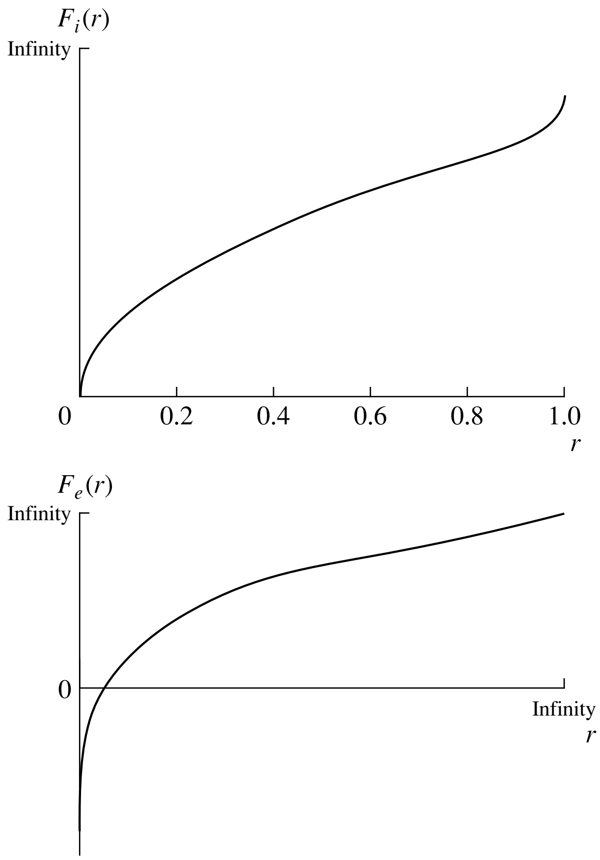
The Hamilton-Jacobi equation can be obtained by making the "new" Hamiltonian K constant ($H \to K \equiv \text{const}$) and, as we know $P_0 \propto K$, it is possible to incorporate this constant into S, and then $S = S(q^\mu, P_i)$. Hereupon, rewriting Eq. (6), we obtain

$$g^{\mu\nu} \frac{\partial Q^0}{\partial x^\mu} \frac{\partial Q^0}{\partial x^\nu} = 1. \tag{15}$$

Partially deriving Eq. (15) with respect to P_i and using Eq. (12b), one obtains

$$g^{\mu\nu} \frac{\partial Q^0}{\partial x^\mu} \frac{\partial Q^i}{\partial x^\nu} = 0. \tag{16}$$

Note that the coordinate system Q^μ together with Eqs. (15) and (16) determine the coordinate transformation relation between a given coordinate system and a Gaussian coordinate system.



The $F(r)$ function for r going from 0 to ∞.

3. APPLICATIONS

This section begins presenting an alternative way of constructing a GCS for the Schwarzschild solution. As we said before, it is very common in the literature (cf. the Novikov coordinates in [4] or unusual ways in [10] and [11]). Then, it follows the main purpose of this paper: a GCS for the Kerr metric. Both of them are constructed by the HJ equation.

3.1. Gaussian System for the Schwarzschild Metric

As a simple exercise, we exhibit a Gaussian coordinate system developed from (1) for the external and internal Schwarzschild solutions. This case is particularly simple due to the symmetries of the metric.

3.1.1. The black-hole case. The Schwarzschild geometry described in the usual coordinate system, for radial observers with nonzero velocity at infinity [10], which was originally given by

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{17}$$

where $r_H = 2M$ and M is the geometric mass of the gravitational source, becomes

$$ds^2 = d\tau^2 - \left(\alpha^2 - 1 + \frac{r_H}{r}\right) dR^2 - r(\tau, R)^2 d\Omega^2, \tag{18}$$

according to the following coordinate transformation:

$$\tau = \alpha t + F(r, \alpha), \quad R \doteq \partial\tau/\partial\alpha, \tag{19}$$

where $\alpha \in \mathbb{R}$ is an arbitrary parameter and τ is interpreted as the proper time of a free particle. Substituting this proposal into Eq. (1), one can see that $F'(r, \alpha) \equiv dF/dr$ must satisfy

$$F'(r, \alpha) = \sqrt{\frac{\alpha^2 - (1 - \frac{r_H}{r})}{(1 - \frac{r_H}{r})^2}}, \tag{20}$$

for the new coordinate system to be admissible. Now, we separate two different regions and change the variable as follows:

$$r > r_H: \quad \left(1 - \frac{r_H}{r}\right) \doteq \alpha^2 \sin^2 x,$$
$$r < r_H: \quad \left(1 - \frac{r_H}{r}\right) \doteq -\alpha^2 \sinh^2 x, \tag{21}$$

and therefore,

$$r > r_H: \quad dF_e = 2r_H \frac{\alpha \cos^2 x}{\sin x(1 - \alpha^2 \sin^2 x)^2} dx,$$
$$r < r_H: \quad dF_i = -2r_H \frac{\alpha \cosh^2 x}{\sinh x(1 + \alpha^2 \sinh^2 x)^2} dx, \tag{22}$$

to be easily integrated giving

$$F_e = \frac{(2\alpha^2 - 1)r_H}{2\sqrt{\alpha^2 - 1}} \ln\left[\frac{\alpha\cos x + \sqrt{\alpha^2 - 1}}{\alpha\cos x - \sqrt{\alpha^2 - 1}}\right] + r_H \frac{\alpha\cos x}{(1 - \alpha^2 \sin^2 x)} + 2\alpha r_H \ln\tan\frac{x}{2},$$
$$F_i = -\frac{(2\alpha^2 - 1)r_H}{2\sqrt{\alpha^2 - 1}} \ln\left[\frac{\alpha\cosh x + \sqrt{\alpha^2 - 1}}{\alpha\cosh x - \sqrt{\alpha^2 - 1}}\right] - r_H \frac{\alpha\cosh x}{(1 + \alpha^2 \sinh^2 x)} - 2\alpha r_H \ln\tanh\frac{x}{2}. \tag{23}$$

The figure illustrates that $F(r)$ is a one-to-one function, thus it well defines a coordinate transformation.

In this general case, the inverse coordinate transformation, $t = t(T, R)$ and $r = r(T, R)$, is given for the r-coordinate by the following implicit expression:

$$\int \frac{dr}{\sqrt{\alpha^2 - 1 + r_H/r}} = -(T - \alpha R). \tag{24}$$

The t-coordinate is also written in an implicit way as a function of T and R. However, a differential relation between them is

$$dt = \frac{\alpha}{(1 - r_H/r)} dT - \frac{(\alpha^2 - 1 + r_H/r)}{1 - r_H/r} dR. \quad (25)$$

Here, we would like to remark that $r = r(T, R)$ is given implicitly in (24). Thus, to calculate $t = t(T, R)$ we must first of all evaluate $r = r(T, R)$. The failure in a search for an analytic expression for the coordinate transformation may seem a problem of the GCS. However, in general relativity calculations, we know that there are derivatives of the metric tensor $g_{\mu\nu}$ which can be evaluated from the original coordinate system making use of the Jacobian matrix. The great advantages of the GCS will appear when we apply it to the quasi-Maxwell equations—cf. Appendix B.

Now, we analyze a particular case in which the inverse coordinate transformation is explicitly expressed. In this case, the mechanical energy is equal to the rest energy of a free particle: $\alpha^2 = 1$ [11], and then we obtain the coordinate transformation by integrating the geodesic equations parameterized by the proper time τ, yielding

$$t = \tau + r_H \left[\ln \left(\frac{\sqrt{r/r_H} + 1}{|\sqrt{r/r_H} - 1|} \right) - 2\sqrt{\frac{r}{r_H}} \right],$$
$$r = \left[-\frac{3}{2} \sqrt{r_H} (\tau + R) \right]^{2/3}. \quad (26)$$

To give a physical interpretation to α, we conclude that $\alpha^2 \geq 1$.

Note that this coordinate transformation does not converge to the inverse coordinate transformation of (19) if we choose $\alpha^2 \to 1$ before the calculations, because we miss the degree of freedom necessary to determine the other coordinates. In both cases we observe that the horizon "disappears", i.e., this coordinate system does not have any problem for $r = r_H$ as in the Eddington-Finkelstein or Kruskal-Szekeres coordinates systems [12]. In these cases, the natural "observers" are null-type. On the other hand, in the Gaussian systems in (19) and (26) the natural observers $V^\mu = \delta^\mu_0$ are timelike at every point, enabling the description of events of the space-time from geodetic massive test particles.

Finally we comment that the Gaussian coordinate system should be mapped into the *Novikov coordinates* (cf. [4]), and then, it will be possible to associate the parameter α to Novikov's parameter R^*. Both coordinate systems should describe the same radial freely falling test particles in the Schwarzschild space-time.

3.1.2. The stellar case. To construct a Gaussian system for the interior part of the Schwarzschild solution, we consider a spherical shell filled with a perfect fluid with energy density $\rho \equiv \rho_0$, where ρ_0 is a constant, co-moving to $u_\mu = (e^{\nu/2}, 0, 0, 0)$. In this case the line element is given by

$$ds^2 = e^{\nu(r)} dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (27)$$

where $e^{\nu(r)}$ is

$$e^{\nu(r)} = \left(\frac{3}{2} \sqrt{1 - r_0^2/r_c^2} - \frac{1}{2} \sqrt{1 - r^2/r_c^2} \right)^2. \quad (28)$$

Here r_0 is the radius of the "star" and $r_c = 3/\rho$. We also assume that $r_0 < r_c$ to avoid singularities in the coordinate system. Similarly to the coordinate transformation (19), we have the following expression for the function $F(r, \alpha)$:

$$F'(r, \alpha) = \sqrt{(\alpha^2 e^{-\nu} - 1)\left(1 - \frac{r_H}{r}\right)}. \quad (29)$$

The differential relation between the two coordinate systems is

$$\begin{pmatrix} dT \\ dR \end{pmatrix} = \begin{pmatrix} \alpha & \sqrt{\alpha^2 - e^{\nu(r)}}/e^{\nu(r)} \\ 1 & -1/\sqrt{\alpha^2 - e^{\nu(r)}} \end{pmatrix} \begin{pmatrix} dt \\ dr \end{pmatrix}. \quad (30)$$

The infinitesimal inverse coordinate transformation is

$$\begin{pmatrix} dt \\ dr \end{pmatrix} = \begin{pmatrix} \alpha/e^{\nu(r)} & -(\alpha^2 - e^{\nu(r)})/e^{\nu(r)} \\ -1/\sqrt{\alpha^2 - e^{\nu(r)}} & \alpha/\sqrt{\alpha^2 - e^{\nu(r)}} \end{pmatrix} \times \begin{pmatrix} dT \\ dR \end{pmatrix}. \quad (31)$$

Again, we claim that to evaluate the t-coordinate, we must calculate the $r = r(T, R)$ function. We note that from Eq. (28), e^ν is a monotonic function, thus it determines the coordinate transformation without ambiguity. With this in mind, the "new" line element written in Gaussian coordinates for the Schwarzschild interior solution is

$$ds^2 = d\tau^2 - (\alpha^2 - e^\nu) dR^2 - r(\tau, R)^2 d\Omega^2. \quad (32)$$

Of course, at $r = r_0$ Eqs. (17) and (27) must be the same. Then, we obtain that $r_H = r_0^3/r_c^2$. The matching conditions for the metric on the stellar surface formulated in the Schwarzschild coordinates are naturally transformed to the Gaussian coordinates.

3.2. The Kerr Metric

In 1963 Roy Kerr found an exact solution [13] of the Einstein equations, which describes the external space-time generated by a "source" provided by a geometric mass M and an angular momentum a per unit of geometric mass. In 1968, Brandon Carter developed a method of obtaining all geodesics of this metric using the relativistic Hamilton-Jacobi equation [14]. In his paper, he considers the principal Hamilton function S as a linear function of the proper time τ of a test particle. Here we identify the two quantities.

In the appendix (see [15]) we exhibit some well-known coordinate systems of the Kerr metric and their main characteristics in our opinion. We include in such list the Gaussian system presented here.

The line element of the Kerr solution in the Boyer-Lindquist coordinates is given by

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 + \frac{4Mra\sin^2\theta}{\rho^2} dt\, d\phi - \left[(r^2+a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right] d\phi^2, \quad (33)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$.

A simple way of calculating the geodesic equations parameterized by the proper time for the Kerr solution is to use the Euler-Lagrange equations encountered in [16]. From the first integrals of these equations we construct the tangent vector field

$$V^0 = \frac{1}{\rho^2\Delta}[\Sigma^2 E - 2MraL],$$
$$V^1 = \frac{F_n(r)}{\rho^2}\Delta,$$
$$V^2 = \frac{G_n(\theta)}{\rho^2},$$
$$V^3 = \frac{1}{\rho^2\Delta}[2Mr(aE - L\csc^2\theta) + \rho^2 L\csc^2\theta], \quad (34)$$

where $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$. In this case, the functions $F_n(r)$ and $G_n(\theta)$ are expressed by

$$F_n(r) = \pm\frac{\sqrt{(E\tilde{r}^2 - aL)^2 - \Delta(r^2+\gamma)}}{\Delta},$$
$$G_n(\theta) = \pm\sqrt{\gamma - a^2\cos^2\theta - \left(Ea\sin\theta - \frac{L}{\sin\theta}\right)^2}, \quad (35)$$

where $\tilde{r}^2 = r^2 + a^2$ and E, L and γ are integration constants.

Let us suppose that the new time coordinate is represented by S, the principal Hamilton function of that problem, and is written in terms of the Boyer-Lindquist coordinates as follows

$$S = -Et + L\phi + W_1(r) + W_2(\theta) \equiv -mQ^0. \quad (36)$$

Substituting Eq. (36) into (15) and assuming $m = 1$ without loss of generality, we obtain

$$E^2\frac{\Sigma^2}{\rho^2\Delta} - \frac{\Delta}{\rho^2}\left(\frac{dW_1}{dr}\right)^2 - \frac{1}{\rho^2}\left(\frac{dW_2}{d\theta}\right)^2 - 2\frac{(2Mr - q^2)}{\rho^2\Delta}EL - \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}L^2 = 1, \quad (37)$$

where E is interpreted as the total mechanical energy and L as the angular momentum of a test particle when we consider the asymptotically flat regime—this interpretation is guaranteed only in such a regime. We can rewrite this expression in a more convenient form as

$$-\Delta\left(\frac{dW_1}{dr}\right)^2 + \frac{(E\tilde{r}^2 - aL)^2}{\Delta} - r^2 = \left(\frac{dW_2}{d\theta}\right)^2 + \left(Ea\sin\theta - \frac{L}{\sin\theta}\right)^2 + a^2\cos^2\theta \equiv \gamma, \quad (38)$$

where γ is a constant of separability. So, we have two equations, one for W_1' and another for W_2':

$$\frac{dW_1}{dr} = \pm\frac{\sqrt{(E\tilde{r}^2 - aL)^2 - \Delta(r^2+\gamma)}}{\Delta},$$
$$\frac{dW_2}{d\theta} = \pm\sqrt{\gamma - a^2\cos^2\theta - \left(Ea\sin\theta - \frac{L}{\sin\theta}\right)^2}. \quad (39)$$

Note that $dW_1/dr = F_n(r)$ and $dW_2/d\theta = G_n(\theta)$.

The other coordinates, which we call $Q^1 \doteq R$, $Q^2 \doteq \Theta$ and $Q^3 \doteq \Phi$, are calculated from the spatial components of the second equation of (12), which yields

$$R \doteq -\frac{\partial S}{\partial E} = t - \frac{\partial W_1}{\partial E} - \frac{\partial W_2}{\partial E},$$
$$\Theta \doteq -\frac{\partial S}{\partial \gamma} = -\frac{\partial W_1}{\partial \gamma} - \frac{\partial W_2}{\partial \gamma},$$
$$\Phi \doteq -\frac{\partial S}{\partial L} = -\phi - \frac{\partial W_1}{\partial L} - \frac{\partial W_2}{\partial L}. \quad (40)$$

The inverse coordinate transformation for an infinitesimal displacement, whose matrix components are written in terms of the vector field (34) for visualization purposes, is

$$\begin{pmatrix} dt \\ dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -V^0 & -V^1 & -V^2 & -V^3 \\ 1 - EV^0 & -EV^1 & -EV^2 & -EV^3 \\ J_{31} & J_{32} & J_{33} & J_{34} \\ -LV^0 & -LV^1 & -L^2 & -(1 + LV^3) \end{pmatrix}$$

$$\times \begin{pmatrix} dT \\ dR \\ d\Theta \\ d\Phi \end{pmatrix}, \qquad (41)$$

where the matrix components J_{31}, J_{32}, J_{33} and J_{34} are given by

$$J_{31} = -2\gamma V^0 + 2a(aE - L) - \frac{4Mra^3 L\cos^2\theta}{\rho^2\Delta},$$
$$J_{32} = -2(\gamma - a^2\cos^2\theta)V^1,$$
$$J_{33} = -2(r^2 + \gamma)V^2,$$
$$J_{34} = -2\gamma V^3 - \frac{2}{\rho^2\Delta}[r^2\Delta(aE + L\csc^2\theta) + a^3(\tilde{r}^2 E - aL) - \tilde{r}^2 a^3 E\sin^2\theta]. \qquad (42)$$

We want to emphasize again that, in spite of the undesirable expression above, to explicitly encounter the coordinates (t, r, θ, ϕ) in terms of (T, R, Θ, Φ)—which is virtually impossible—we always need the Jacobian matrix only, or its inverse, to do all calculations in GR. For instance, the remaining metric components can be expressed in the form

$$\bar{g}_{11} = 1 - \frac{2Mr}{\rho^2} - E^2,$$
$$\bar{g}_{12} = -2[E(\gamma - a^2) + aL],$$
$$\bar{g}_{13} = -EL - \frac{2Mar\sin^2\theta}{\rho^2},$$
$$\bar{g}_{22} = -4(r^2 + \gamma)(\gamma - a^2\cos^2\theta),$$
$$\bar{g}_{23} = -2\left[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a\sin^2\theta\right],$$
$$\bar{g}_{33} = -(L^2 + \tilde{r}^2\sin^2\theta) - \frac{2Mra^2\sin^4\theta}{\rho^2}. \qquad (43)$$

The determinant of this new metric $\bar{g}_{\mu\nu}$ is

$$\bar{g} \doteq \det \bar{g}_{\mu\nu} = -4f(r)h(\theta), \qquad (44)$$

where

$$f(r) \doteq (E\tilde{r}^2 - aL)^2 - \Delta(r^2 + \gamma),$$
$$h(\theta) \doteq (\gamma - a^2\cos^2\theta)\sin^2\theta - (Ea\sin^2\theta - L)^2.$$

Then the metric $\bar{g}_{\mu\nu}$ is well-defined for these events if we guarantee that $f, h > 0$.

The usual coordinate systems for the Kerr metric present mathematical complications in some surfaces which are called *horizons*. In the Boyer-Lindquist coordinates, for example, the mathematical expression for the Kerr horizons is

$$r_\pm = M \pm \sqrt{M^2 - a^2} \qquad (45)$$

where $r_{+/-}$ is called the outer/inner horizon. Differently from the other coordinate systems, the metric components expressed in the Gaussian system showed before are completely regular at the horizons. Note that a divergence still exists at the real singularity $r = 0$ and $\theta = \pi/2$, but just at this point.

Now let us analyze a specific set of observer fields. We choose the geodetic field in GCS $\bar{V}^\mu \doteq \delta^\mu_0$. Taking the inverse coordinate transformation, we write this observer field in Kerr coordinates in the following way:

$$V^0 = \frac{1}{\rho^2\Delta}[\Sigma^2 E - 2MraL],$$
$$V^1 = \frac{W_1'}{\rho^2}\Delta,$$
$$V^2 = \frac{W_2'}{\rho^2},$$
$$V^3 = \frac{1}{\rho^2\Delta}[2Mr(aE - L\csc^2\theta) + \rho^2 L\csc^2\theta]. \qquad (46)$$

If we compare Eq. (46) with (34), which is obtained from the Euler-Lagrange equations, we conclude that they are the same. Therefore, δ^μ_0 corresponds to all timelike geodetic vectors of the Kerr solution.

4. INCOMPLETENESS OF GAUSSIAN SYSTEMS AND RELATED QUESTIONS

Up to now, there is no theorem that specifies the necessary and sufficient conditions for the existence of a complete Gaussian coordinate system for an arbitrary metric or whether the hyper-surface defined by the Gaussian vector field is a Cauchy surface [18].

In our case, the impossibility to identify S with a Cauchy surface due to the presence of closed timelike curves in the maximally extended Kerr manifold is obvious. Nevertheless, what can be said about the Gaussian observers and their class of parameters when we think about information theory? Can we do thermodynamics with Gaussian observers? A concrete answer to these questions is very difficult. Nowadays, still are found controversial arguments [19] whenever this theme emerges. It seems that the only "good" observers to do thermodynamics are the accelerated ones, due to the presence of unaccessible regions of the manifold for such observers (see *Rindler observers* in [20]).

In the meantime, one can analyze occasional divergences in the expansion factor $\vartheta \doteq V^\mu{}_{;\mu}$ of the observer field $V^\mu = \delta^0_\mu$. In a specific coordinate system, it can help us to recognize the regions where a given congruence of observers remains well-defined.

As we know, the observer field V_μ is geodesic and the gradient of the hypersurface T. Therefore, the vorticity $\omega_{\mu\nu} \equiv h^{\alpha}_{\mu} h^{\beta}_{\nu}(V_{\alpha;\beta} - V_{\beta;\alpha})$ and the acceleration $a^\mu \equiv V^{\mu}{}_{;\nu}V^\nu$ are zero. The shear tensor $\sigma_{\mu\nu} \equiv h^{\alpha}_{\mu} h^{\beta}_{\nu}(V_{\alpha;\beta} + V_{\beta;\alpha})$, which is also an important kinematical property, will be out of our qualitative analysis because we get here satisfactory enough results using only the expansion factor.

A congruence of observers is understood as a given choice of parameters E, L and γ. The explicit expression for the expansion, in this case, is

$$\vartheta = \frac{\cos\theta}{\rho^2\sqrt{h}}\big[\gamma - a^2\big(\cos^2\theta + (2E^2 - 1)\sin^2\theta\big) + 2EaL\big] - \frac{1}{\rho^2\sqrt{f}}\big[2Er(E\tilde{r}^2 - aL) - r\Delta - (r^2+\gamma)(r - M)\big], \tag{47}$$

where the functions f and h were defined in Sec. 3.2. We define $\mathcal{Q} \doteq \gamma - (Ea - L)^2$ and choose a specific set of congruences such that $\mathcal{Q} = 0$. If one approaches the singularity ($r = 0$ and $\theta = \pi/2$), from positive values of r, we obtain that $\vartheta \to -\infty$, completely independent of θ. From this result, we conclude that these congruences cannot reach the singularity, because the expansion factor is not well defined there. On the other hand, we see that this set is the only well-defined one for the whole Kerr external ($r > 0$) space-time. In other words, these geodesic congruences are the unique ones that cover the Kerr space-time region with intuitive physical meaning, allowing us to interpret, without ambiguities, E as the specific energy of a test particle, L as the angular momentum and γ as being a first integral system coming from a specific combination of the momenta p_r and p_θ (remember that as physical objects, we assume that we are in the region $r \to \infty$).

If we pick out a given black hole with mass $M = 2$ and angular momentum $a^2 = 1$ (these numbers are completely arbitrary and do not interfere in the results, but respecting the relation $M^2 > a^2$), we can make a more detailed analysis controlling the boundaries of functions $f(r)$ and $h(\theta)$. Making a change of variables $x = \sin^2\theta$ in $h(\theta)$, we get

$$h(x) = (1 - E^2)x^2 + (\gamma - 1 + 2EL)x - L^2. \tag{48}$$

From the roots of the polynomial function above, we can single out intervals in the range of the parameters, covering all values of θ. For instance, we can choose $L = 0$ and $E^2 > 1$, and consequently we obtain $E^2 \le \gamma$. On the other hand, the $f(r)$ function becomes

$$f(r) = (E^2 - 1)r^4 + 4r^3 + (2E^2 - 1 - \gamma)r^2 + 4\gamma r + (E^2 - \gamma). \tag{49}$$

From the 0-order term of the polynomial in r, we see that $E^2 \ge \gamma$ is a necessary condition for $f(r)$ be greater than 0 for all values of r. From this, we conclude that it is impossible to cover all manifold events with only one congruence of observers. Just for comparison, the same thing happens in the Gödel metric, as we can see in [7].

A very important remark is that it works like a chronological protection [21] for free test particles coming from infinity if we assume that this type of observers get information only from their synchronized "colleagues", while the values $r < 0$—where we expect to encounter CTC's for the Kerr space-time—are forbidden for them. With this in mind, we are left to establish some connection between the domain of validity of a given geodesic congruence and the CTC's. However, the mathematical relation between them is still obscure for us.

5. GENERALIZATION (KERR–NEWMAN METRIC)

In 1965 E. Newman et al. [17] found a generalization of Kerr's solution, which besides describing a black hole with geometrical mass M and angular momentum a per unit of geometrical mass, includes a charge q.

The line element for the Kerr-Newman solution is

$$ds^2 = \left(1 - \frac{(2Mr - q^2)}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 + \frac{2(2Mr - q^2)a\sin^2\theta}{\rho^2}dtd\phi - \bigg[(r^2 + a^2)\sin^2\theta + \frac{(2Mr - q^2)a^2\sin^4\theta}{\rho^2}\bigg]d\phi^2, \tag{50}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 + q^2 - 2Mr$. If one follows step by step the method in the previous sections (the Schwarzschild and Kerr cases), at the end one shall find the covariant metric components of the Kerr-Newman solution described in a GCS, as follows:

$$\bar{g}_{11} = 1 - \frac{2Mr - q^2}{\rho^2} - E^2,$$
$$\bar{g}_{12} = -2[E(\gamma - a^2) + aL],$$
$$\bar{g}_{13} = -EL - \frac{(2Mr - q^2)a\sin^2\theta}{\rho^2},$$
$$\bar{g}_{22} = -4(r^2 + \gamma)(\gamma - a^2\cos^2\theta),$$
$$\bar{g}_{23} = -2\left[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a\sin^2\theta\right],$$
$$\bar{g}_{33} = -(L^2 + \tilde{r}^2\sin^2\theta)$$

$$- \frac{(2Mr - q^2)a^2 \sin^4\theta}{\rho^2}. \tag{51}$$

The vector field $\bar{V}^\mu \doteq \delta^\mu_0$ also corresponds to all time-like geodesic vectors of the Kerr-Newman solution.

If we take appropriate limits, we get GCS's for other metrics: in the case where $a = 0$ and $q \neq 0$, we obtain a Gaussian system for the Reissner-Nordström solution. If $a \neq 0$ and $q = 0$, we get the previous Gaussian system for the Kerr metric constructed before. Finally, if $a = q = 0$ it yields a Gaussian system for the Schwarzschild case.

6. CONCLUSIONS

In this paper, we have built a set of Gaussian coordinate systems for the Kerr metric and generalizations, containing the whole set of physically relevant black holes.

From the analysis of incompleteness of the Gaussian system, we conclude that the domain of validity of a given congruence of observers is intrinsically related to the conserved quantities of a test particle. This result suggests a possible connection between the GCS and the chronological protection conjecture. This possible connection is also sustained by the fact that the vorticity $\omega_{\mu\nu}$ and the magnetic part of the Weyl tensor $H_{\alpha\beta} \doteq -{}^*W_{\alpha\mu\beta\nu}V^\mu V^\nu$ are identically zero for Gaussian observers.

Gaussian observers for the Kerr metric describe the metric structure of the space-time as time-dependent. This time dependence can be exhibited from measurements realized by Gaussian observers in their neighborhoods. This important feature seems to single out Gaussian observers in the description of the physical world.

Appendix A

SOME SPECIAL COORDINATE SYSTEMS FOR THE KERR METRIC

As we said, in [15] we see many coordinate systems for the Kerr metric encountered in the literature and the coordinate transformations between them. Here is a list with some properties and the inclusion of our Gaussian system.

Kerr's original coordinates. In this coordinate system (u, r, θ, ϕ), the line element is

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) du^2 - \rho^2 d\theta^2 - 2a\sin^2\theta dr d\phi$$

$$- 2du dr - \frac{4Mra\sin^2\theta}{\rho^2} du\, d\phi$$

$$- \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right] d\phi^2, \tag{A1}$$

where $\rho^2 \equiv r^2 + a^2\cos^2\theta$.

The most important features in this coordinate system are:

- The appearance of an actual singularity in g_{uu}, for $r = 0$ and $\theta = \pi/2$;
- Setting $a \to 0$, the line element reduces to the Schwarzschild geometry in the "advanced Eddington-Finkelstein coordinates";
- In terms of M, the line element can be put into a Kerr-Schild form by $g^{\mu\nu} = g^{\mu\nu}_0 + f(M, a, x^\alpha) l^\mu l^\nu$, where $g^{\mu\nu}_0$ is the Minkowski metric and l^μ is a geodetic null vector.

The Kerr-Schild "Cartesian" coordinates. By use of (x^0, x, y, z) coordinates, ds^2 for the Kerr metric becomes

$$ds^2 = (dx^0)^2 - dx^2 - dy^2 - dz^2$$

$$- \frac{2Mr^3}{r^4 + a^2z^2}\Big[dx^0 + \frac{r}{a^2 + r^2}(x\, dx + y\, dy)$$

$$+ \frac{a}{a^2 + r^2}(y\, dx - x\, dy) + \frac{z}{r}dz\Big]^2. \tag{A2}$$

The main characteristics of this coordinate system are:

- For $M \to 0$, it is the Minkowski space-time in Cartesian coordinates;
- For $a \to 0$, it is the Schwarzschild solution in Cartesian coordinates;
- The full metric $(M, a \neq 0)$ is obviously the Kerr-Schild form again.

The Boyer-Lindquist coordinates. The most useful coordinate system for the Kerr metric are the Boyer-Lindquist coordinates, so that

$$ds^2 = \left(1 - \frac{2M\,r}{\rho^2}\right) d\hat{t}^2 - \frac{\rho^2}{\Delta}\, dr^2 - \rho^2 d\theta^2$$

$$- \left(r^2 + a^2 + \frac{2Ma\,r}{\rho^2}\, \sin^2\theta\right) \sin^2\theta\, d\hat{\phi}^2$$

$$+ \frac{4Ma\,r\, \sin^2\theta}{\rho^2} d\hat{t}\, d\hat{\phi}, \tag{A3}$$

where $\Delta \equiv r^2 + a^2 - 2Mr$.

The noticeable properties are that:

- It minimizes the number of off-diagonal components of the metric;

- The asymptotic behavior of these coordinates permits to conclude that M is indeed the mass and $J = Ma$ is the angular momentum;
- For $a \to 0$, it is the Schwarzschild solution in the standard coordinate system;
- For $M \to 0$, it is the Minkowski line element in oblate spheroidal coordinates;
- It is a maximal extension of the Kerr manifold.

Rational polynomial coordinates. If we make a coordinate transformation $\chi = \cos\theta$ from the Boyer-Lindquist coordinates, we have the following new version of the Kerr spacetime:

$$ds^2 = \left(1 - \frac{2M\,r}{r^2 + a^2\chi^2}\right) d\hat{t}^2 - \frac{r^2 + a^2\chi^2}{\Delta}\, dr^2 - \frac{(r^2 + a^2\chi^2)}{1 - \chi^2} d\chi^2 + \frac{4Ma\,r(1-\chi^2)}{r^2 + a^2\chi^2} d\hat{t} d\hat{\phi} - (1 - \chi^2) \times \left(r^2 + a^2 + \frac{2Ma\,r(1-\chi^2)}{r^2 + a^2\chi^2}\right) d\hat{\phi}^2. \quad \text{(A4)}$$

These coordinates possess the following qualities:

- All metric components are rational polynomials of the coordinates;
- The absence of trigonometric functions makes computational calculations faster than in other coordinate systems.

The Doran coordinates. Introduced by C. Doran in 2000, here we obtain another coordinate system for Kerr metric given by

$$ds^2 = dt^2 - \rho^2 d\theta^2 - (r^2 + a^2)\sin^2\theta d\phi^2 - \frac{\rho^2}{r^2 + a^2}\left[dr + \frac{\sqrt{2Mr(r^2 + a^2)}}{\rho^2} \times (dt - a\sin^2\theta d\phi)\right]^2. \quad \text{(A5)}$$

Useful features of the Doran coordinates are:

- For $a \to 0$, it is the Schwarzschild geometry in the Painlevé–Gullstrand form;
- For $M \to 0$, we obtain the Minkowski metric in oblate spheroidal coordinates;
- In Doran's coordinates, the contravariant metric component g^{00} is equal to 1;
- According to the ADM formalism, Doran's coordinates slice the Kerr metric so that the "lapse" function is 1 everywhere.

The Gaussian coordinates. Constructed from the relativistic Hamilton-Jacobi equation, the line element for the Kerr Metric in a Gaussian coordinate system is

$$ds^2 = dT^2 - \left(E^2 - 1 + \frac{2Mr}{\rho^2}\right) dR^2 - 4(r^2 + \gamma)(\gamma - a^2\cos^2\theta)d\Theta^2 - \left(L^2 + \tilde{r}^2\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right) d\Phi^2 - 2\left[E(\gamma - a^2) + aL\right] dR\, d\Theta - \left(EL + \frac{2Mar\sin^2\theta}{\rho^2}\right) dR\, d\Phi - 2\Big[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a\sin^2\theta\Big] d\Theta\, d\Phi, \quad \text{(A6)}$$

where $\tilde{r} = r^2 + a^2$.

Besides the usual characteristics of Gaussian coordinates, we list:

- For massive test particles, the geodesic equations parameterized by the proper time are immediately integrated;
- Differently from all other cases, the metric is proper time (T)-dependent;
- This metric depends on the parameters of the observer field comoving to the reference frame.

Appendix B

THE SCHWARZSCHILD SOLUTION FROM QUASI-MAXWELL EQUATIONS WRITTEN IN GAUSSIAN COORDINATES

This appendix has relevance on the way of searching for an internal solution for the Kerr metric. Although we get a negative result, we present an alternative formalism, which considerably simplifies the equations, but introduces a larger set of coupled nonlinear equations than Einstein's ones.

It was proved in [22] that the JEK (Jordan-Ehlers-Kundt) equations are equivalent to those of GR. Besides, they become particularly simple when expressed in a Gaussian coordinate system. In principle, we could use this simplicity to search for an internal solution for the Kerr metric. As an example, we apply this method to obtain the stellar Schwarzschild solution, as follows.

There are many references addressing the formal deduction of these equations and their properties, for instance, in [23]. These equations (JEK) can be obtained from Bianchi's identities together with the Einstein equations:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2}T^{\mu[\alpha;\beta]} + \frac{1}{6}g^{\mu[\alpha}T^{,\beta]}. \tag{B1}$$

From this we can obtain four independent projections of the Weyl tensor divergence:

$$W^{\alpha\beta\mu\nu}{}_{;\nu}V_\beta V_\mu h_\alpha{}^\sigma, \tag{B2}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\eta^{\sigma\lambda}{}_{\alpha\beta}V_\mu V_\lambda, \tag{B3}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}h_\mu{}^{(\sigma}\eta^{\tau)\lambda}{}_{\alpha\beta}V_\lambda, \tag{B4}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}V_\beta h_{\mu(\tau}h_{\sigma)\alpha}. \tag{B5}$$

Besides, we also have the conservation law for the energy-momentum tensor

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{B6}$$

which can be projected parallel to an observer field V^μ or perpendicular to it, as follows:

$$T^{\mu\nu}{}_{;\nu}V^\mu = 0, \tag{B7}$$

$$T^{\mu\nu}{}_{;\nu}h^{\mu\alpha} = 0, \tag{B8}$$

where $h^{\mu\nu} \doteq g^{\mu\nu} - V^\mu V^\nu$.

From the Riemann tensor, we can find the evolution equations for the kinematical quantities (*expansion* ϑ, *shear* $\sigma_{\mu\nu}$ and *vorticity* $\omega_{\mu\nu}$). They obey the relations

$$\dot{\vartheta} + \frac{\vartheta^2}{3} + 2(\sigma^2+\omega^2) - a^\alpha{}_{;\alpha} = R_{\mu\nu}V^\mu V^\nu, \tag{B9}$$

$$\begin{aligned} &h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \frac{1}{3}h_{\alpha\beta}(-2(\sigma^2+\omega^2) + a^\lambda{}_{;\lambda}) \\ &\quad + a_\alpha a_\beta - \frac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu(a_{\mu;\nu} + a_{\nu;\mu}) \\ &\quad + \frac{2}{3}\vartheta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^\mu{}_\beta + \omega_{\alpha\mu}\omega^\mu{}_\beta \\ &= R_{\alpha\epsilon\beta\nu}V^\epsilon V^\nu - \frac{1}{3}R_{\mu\nu}V^\mu V^\nu h_{\alpha\beta}, \end{aligned} \tag{B10}$$

$$\begin{aligned} &h_\alpha{}^\mu h_\beta{}^\nu \dot{\omega}_{\mu\nu} - \frac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu(a_{\mu;\nu} - a_{\nu;\mu}) \\ &+ \frac{2}{3}\vartheta\omega_{\alpha\beta} - \sigma_{\beta\mu}\omega^\mu{}_\alpha + \sigma_{\alpha\mu}\omega^\mu{}_\beta = 0, \end{aligned} \tag{B11}$$

together with the constraint equations

$$\begin{aligned} R_{\mu\nu}V^\mu h^\nu{}_\lambda = \frac{2}{3}\vartheta_{,\mu}h^\mu{}_\lambda - (\sigma^\alpha{}_\gamma + \omega^\alpha{}_\gamma)_{;\alpha}h^\gamma{}_\lambda \\ - a^\nu(\sigma_{\lambda\nu} + \omega_{\lambda\nu}), \end{aligned} \tag{B12}$$

$$\omega^\alpha{}_{;\alpha} + 2\omega^\alpha a_\alpha = 0, \tag{B13}$$

$$\begin{aligned} H_{\tau\lambda} = -\frac{1}{2}h_{(\tau}{}^\epsilon h_{\lambda)}{}^\alpha \eta_\epsilon{}^{\beta\gamma\nu}V_\nu(\sigma_{\alpha\beta} + \omega_{\alpha\beta})_{;\gamma} \\ + a_{(\tau}\omega_{\lambda)}. \end{aligned} \tag{B14}$$

If we assume that Einstein's equations are only valid on a Cauchy surface Σ, the set of equations (B2)–(B14), the so-called *quasi-Maxwell equations*, propagate GR solutions from Σ to the whole space-time.

Let us consider a diagonal metric, similar to the Schwarzschild one described in a GCS:

$$ds^2 = dT^2 - B(T,R)dR^2 - r^2(T,R)d\Omega^2, \tag{B15}$$

and an observer field $V^\mu \doteq \delta^\mu_0$. The expansion ϑ for this vector field is

$$\vartheta = \frac{1}{2}\left(\frac{\dot{B}}{B} + \frac{4\dot{r}}{r}\right), \tag{B16}$$

where $\dot{Y}(T,R) \doteq \partial Y/\partial T$. After that, we calculate the shear tensor $\sigma^\mu{}_\nu$ and the electric part of the Weyl tensor ($E^\mu{}_\nu \doteq -W_{\alpha\mu\beta\nu}V^\mu V^\nu$) and write them in the matrix form

$$[\sigma^i{}_j] = f(T,R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \tag{B17}$$

$$[E^i{}_j] = g(T,R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \tag{B18}$$

where

$$f(T,R) = \frac{1}{3}\left(\frac{\dot{B}}{B} - \frac{2\dot{r}}{r}\right) \tag{B19}$$

and

$$\begin{aligned} g(T,R) = \frac{1}{12r^2B^2}(-2r^2B\ddot{B} + r^2\dot{B}^2 - 4rBr'' \\ + 2rB\dot{r}\dot{B} + 2rr'B' + 4rB^2\ddot{r} - 4B^2 \\ - 4B^2\dot{r}^2 + 4Br'^2), \end{aligned} \tag{B20}$$

where $Y'(T,R) \doteq \partial Y/\partial R$.

Observe that both $\sigma_{\mu\nu}$ and $E_{\mu\nu}$ are proportional to the same matrix. It will quite simplify the matrix equations to scalar equations. All other quantities such as the magnetic part of the Weyl tensor $H_{\alpha\beta}$, the vorticity $\omega_{\alpha\beta}$ and the acceleration ($a^\mu \doteq V^\mu{}_{;\nu}V^\nu$) are identically zero due to properties of the observer congruence chosen.

Let us assume that $V^\mu = \delta^\mu_0$ in Gaussian coordinates is co-moving to an arbitrary fluid, which can be expressed by

$$T_{\mu\nu} = (\rho+p)V_\mu V_\nu - pg_{\mu\nu} + q_{(\mu}V_{\nu)} + \pi_{\mu\nu}, \tag{B21}$$

where ρ is the energy density, p is the isotropic pressure, q_μ is the heat flux and $\pi_{\mu\nu}$ is the anisotropic pressure.

In the case of the Schwarzschild stellar solution, it is assumed to be a perfect fluid inside a spherical shell and an accelerated observer ($u_\mu = \sqrt{g_{00}}\,\delta^0_\mu$) comoving to this fluid. With respect to the Gaussian observers, such fluid presents a heat flux $q^\mu = (0, q^1, 0, 0)$ and an anisotropic pressure

$$[\pi^i{}_j] = \pi(T,R)\begin{pmatrix}1 & 0 & 0\\ 0 & -1/2 & 0\\ 0 & 0 & -1/2\end{pmatrix}. \tag{B22}$$

With these considerations, the set of equations (B2)–(B5) takes the form

$$g' + 3\frac{r'}{r}g = \frac{1}{3}\rho' + \frac{\dot r}{r}q_1 + \frac{1}{2}\left(\pi' + 3\frac{r'}{r}\pi\right), \tag{B23}$$

$$\dot g + 3\frac{\dot r}{r}g = \frac{1}{4}f\pi - \frac{1}{2}(\rho+p)f + \frac{1}{2}\dot\pi + \frac{1}{6}\vartheta\pi - \frac{1}{3}\left[(q^1)' + \left(\frac{1}{2}\frac{B'}{B} - 2\frac{r'}{r}\right)q^1\right]. \tag{B24}$$

The conservation laws (B7), (B8) can be written explicitly as

$$\dot\rho + (\rho+p)\vartheta - \frac{3}{2}f\pi + (q^1)' + \left(\frac{1}{2}\frac{B'}{B} + 2\frac{r'}{r}\right) = 0, \tag{B25}$$

$$\pi' + 3\frac{r'}{r}\pi + q_{1,0} + \vartheta q_1 - p' = 0. \tag{B26}$$

The evolution equations of the kinematical quantities are the following:

$$\dot\vartheta + \frac{\vartheta^2}{3} + \frac{3}{2}f^2 = -\frac{1}{2}(\rho+3p), \tag{B27}$$

$$\dot f + \frac{f^2}{2} + \frac{2}{3}\vartheta f = -g - \frac{1}{2}\pi^1{}_1. \tag{B28}$$

Finally, the remaining constraint equation is

$$f' + 3\frac{r'}{r}f - \frac{2}{3}\vartheta' = 0. \tag{B29}$$

Hereupon the set of equations (B23)–(B29) corresponds to the problem of initial conditions which shall give origin to Schwarzschild solution, according to the Birkhoff theorem. As it is not our aim, we will not solve these equations here. However, we will indicate how to proceed (see details in [10]).

First, we analyze the Einstein equations for $T_{\mu\nu} = 0$, and we obviously obtain $B(T,R)$ and $r(T,R)$ like those given in Eq. (18), identifying $T = \tau$. On the other hand, from the quasi-Maxwell equations we get

$$B = \frac{r'^2}{1 + h(R)}, \tag{B30}$$

$$\dot r = \sqrt{y(R) + k/r}, \tag{B31}$$

$$r' = b(R)\sqrt{h(R) + k/r}, \tag{B32}$$

where $h(R)$, $y(R)$ and $b(R)$ are arbitrary functions and k is a constant. If we assume as the initial condition surface $r(T,R) \equiv const \to \infty$, then we will obtain the Schwarzschild external solution.

The Schwarzschild internal solution can be obtained if we consider the energy-momentum tensor associated with the Gaussian observer δ^μ_0, written in terms of the quantities associated with the observer u_μ (the energy density ρ and pressure p) presented in Sec. 3.1.2, as follows:

$$\rho_G = (\rho+p)\alpha^2 e^{-\nu} - p, \tag{B33}$$

$$p_G = -\frac{1}{3}[(\rho+p)(1-\alpha^2 e^{-\nu}) - 3p], \tag{B34}$$

$$q^\mu = (\rho+p)\alpha e^{-\nu}(0,1,0,0), \tag{B35}$$

$$\pi = \frac{2}{3}(1 - \alpha^2 e^{-\nu}) \tag{B36}$$

where $\nu = \nu(T,R)$ and α is an external parameter. Substituting these equations into the quasi-Maxwell equations, we will find exactly the Schwarzschild internal solution with some arbitrary functions. However, we must match this solution with that coming from the initial condition, that is, Einstein's equations on the hypersurface. Besides, choosing the Cauchy surface as $r(T,R) = r_0$, we fix the arbitrary functions by obtaining the Schwarzschild stellar solution with that procedure.

Although there are already many contributions to the subject—cf. [24, 25] and [26]—from this example, we conclude that the search for an internal solution for the Kerr metric should not be reduced to a perfect-fluid solution in the Gaussian coordinate system. We expect that the associated observers detect, for instance, a heat flux as in the case of spherically symmetric metrics above. The heat flux appears naturally from the Einstein-Hilbert action if we add some Lagrange multipliers ρ_0 and q_i corresponding to the choice of the Gaussian coordinates system *a priori*

$$S = \frac{1}{2}\int\sqrt{-g}\,d^4x \times [R + 2\rho_0(g^{00} - 1) + 2q_i g^{0i}]. \tag{B37}$$

The variational principle with respect to ρ_0 and q_i fixes the coordinate system, whereas the variational principle with respect to $g_{\mu\nu}$ produces some extra degrees of freedom corresponding to the energy density

ρ_0 and the heat flux q_i, due to the definition of the energy-momentum tensor

$$T_{\mu\nu} \doteq \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}. \qquad \text{(B38)}$$

It is clear that the quasi-Maxwell equations for the Kerr metric in the Gaussian system is rather more involved than in the Schwarzschild case. So, one could be led to modify the Kerr metric in a GCS to another GCS with simpler metric components, in order to find an internal solution for the Kerr metric making use of non-perfect fluids. This work is under preparation.

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