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EFFECTIVE GEOMETRY IN NONLINEAR ELECTRODYNAMICS

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The electromagnetic force a photon undergoes in a nonlinear regime can be geometrized. This is a rather unexpected result and at the same time a beautiful consequence of the analysis of the behavior of the discontinuities of non-homogeneous nonlinear electromagnetic field. We show how such geometrization is possible.

Keywords: Nonlinear electrodynamics; geometrization.

1. General Comments on Nonlinear Electrodynamics

Modifications of light propagation in different vacua states has recently been a subject of interest. Such investigation shows that, under distinct non trivial vacua (related to several circumstances such as temperature effects, particular boundary conditions, quantum polarization, etc), the motion of light can be viewed as electromagnetic waves propagating through a classical dispersive medium. The medium induces modifications on the equations of motion, which are described in terms of nonlinearities of the field. In order to apply such a *medium interpretation* we consider modifications of electrodynamics due to virtual pair creation. In this case the effects can be simulated by an effective Lagrangian which depends only on the two gauge invariants F and G of the electromagnetic field.[1]

One of the main achievements of such investigation is the understanding that, in such nonlinear framework, *photons* propagate along geodesics that are no more null in the actual Minkowski spacetime but in another effective geometry. Although the basic understanding of this fact — at least for the specific case of Born-Infeld electrodynamics — has been known for a long time,[1] it has been scarcely noticed in the literature. Moreover, its consequences were not exploited any further. In particular, we emphasize the general application and the corresponding consequences of the method of the effective geometry outlined here.

The exam of the photon propagation beyond Maxwell electrodynamics has a rather diversified history: it has been investigated in curved spacetime, as a consequence of non-minimal coupling of electrodynamics with gravity and in nontrivial QED vacua, as an effective modification induced by quantum fluctuations.[1] As a consequence of this examination some unexpected results appear. Just to point one out, we mention the possibility of *faster-than-light* photons.[a]

The general approach of all these theories is based on a gauge invariant effective action, which takes into account modifications of Maxwell electrodynamics induced by different sorts of processes. Such a procedure is intended to deal with the quantum vacuum as if it is a classical medium. Another important consequence of such point of view is the possibility to interpret all such vacua modifications — with respect to the photon propagation — as an effective change of the spacetime metric properties. This result allows one to appeal to an analogy with the electromagnetic wave propagation in curved spacetime due to gravitational phenomena.

1.1. *Definitions and notations*

We call the electromagnetic tensor $F_{\mu\nu}$, while its dual $F^*_{\mu\nu}$ is

$$F^*_{\alpha\beta} \doteq \frac{1}{2}\eta_{\alpha\beta}{}^{\mu\nu}F_{\mu\nu}, \tag{1}$$

where $\eta_{\alpha\beta\mu\nu}$ is the completely antisymmetric Levi-Civita tensor; the Minkowski metric tensor is represented by its standard form $\eta^{\mu\nu}$. The two invariants constructed with these tensors are defined as $F \doteq F^{\mu\nu}\,F_{\mu\nu}$, $G \doteq F^{\mu\nu}\,F^*_{\mu\nu}$.

Once the modifications of the vacuum which will be dealt here do not break the gauge invariance of the theory, the general form of the modified Lagrangian for electrodynamics may be written as a functional of the above invariants, that is, $L = L(F, G)$. We denote by L_F and L_G the derivatives of the Lagrangian L with respect to the invariant F and G, respectively; and similarly for the higher order derivatives. We are particularly interested in the derivation of the characteristic surfaces which guide the propagation of the field discontinuities.

Let Σ be a surface of discontinuity for the electromagnetic field. Following Hadamard we assume that the field itself is continuous when crossing Σ, while its first derivative presents a finite discontinuity. We accordingly set

$$[F_{\mu\nu}]_\Sigma = 0, \quad [\partial_\lambda\, F_{\mu\nu}]_\Sigma = f_{\mu\nu}k_\lambda, \tag{2}$$

in which the symbol $[J]_\Sigma \equiv \lim_{\delta\to 0^+}(J|_{\Sigma+\delta} - J|_{\Sigma-\delta})$ represents the discontinuity of the arbitrary function J through the surface Σ characterized by the equation $\Sigma(x^\mu) = constant$. The tensor $f_{\mu\nu}$ is called the discontinuity of the field, and $k_\lambda = \partial_\lambda\,\Sigma$ is the propagation vector.

2. The Method of the Effective Geometry

2.1. *One-parameter Lagrangians*

In this section we will investigate the effects of nonlinearities in the equation of evolution of electromagnetic waves. We consider in the first part to the simple class

[a]The meaning of such expression is that the wave propagates along spacelike characteristics in the Minkowski background.

of gauge invariant Lagrangians defined by $L = L(F)$. From the least action principle we obtain the field equation

$$\partial_\mu \left(L_F F^{\mu\nu}\right) = 0. \tag{3}$$

Applying conditions (2) for the discontinuity of the field equation (3) through Σ we obtain

$$L_F f^{\mu\nu} k_\nu + 2L_{FF}\,\xi F^{\mu\nu} k_\nu = 0, \tag{4}$$

where $\xi \doteq F^{\alpha\beta} f_{\alpha\beta}$. The consequence of discontinuity in the cyclic identity is

$$f_{\mu\nu} k_\lambda + f_{\nu\lambda} k_\mu + f_{\lambda\mu} k_\nu = 0. \tag{5}$$

In order to obtain a scalar relation we contract this equation with $k_\alpha \eta^{\alpha\lambda} F^{\mu\nu}$:

$$\xi k_\nu k_\mu \eta^{\mu\nu} + 2F^{\mu\nu} f_\nu{}^\lambda k_\lambda k_\mu = 0. \tag{6}$$

Let us consider the case in which ξ does not vanish.[b] From equations (4) and (6) we obtain the propagation equation for the field discontinuities as given by

$$\left(L_F \eta^{\mu\nu} - 4L_{FF} F^{\mu\alpha} F_\alpha{}^\nu\right) k_\mu k_\nu = 0. \tag{7}$$

Expression (7) suggests that one can interpret the self-interaction of the background field $F^{\mu\nu}$, in what concerns the propagation of electromagnetic discontinuities, as if it had induced a modification on the spacetime metric $\eta_{\mu\nu}$, leading to the effective geometry

$$g^{\mu\nu}_{\text{eff}} = L_F\,\eta^{\mu\nu} - 4\,L_{FF}\,F^\mu{}_\alpha\,F^{\alpha\nu}. \tag{8}$$

A simple inspection of this equation shows that only in the particular case of linear Maxwell electrodynamics the discontinuity of the electromagnetic field propagates along null paths in the Minkowski background.

The general expression of the effective geometry can be equivalently written in terms of the vacuum expectation value (VEV) of the energy-momentum tensor

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-\gamma}}\,\frac{\delta\,\Gamma}{\delta\,\gamma^{\mu\nu}}, \tag{9}$$

where Γ is the effective action and $\gamma_{\mu\nu}$ the is Minkowski metric written in an arbitrary coordinate system (γ is the corresponding determinant). In the case of one-parameter Lagrangians we obtain

$$T_{\mu\nu} = -4L_F\,F_\mu{}^\alpha\,F_{\alpha\nu} - L\,\eta_{\mu\nu}, \tag{10}$$

[b]For the case in which $\xi = 0$, the quantity $f_{\mu\nu}$ is a singular two-form. Following Lichnerowicz it can be decomposed in terms of the propagating vector k_μ and a spacelike vector $a_\mu = a\,\epsilon_\mu$ orthogonal to k_μ, in which ϵ_μ is the normalized polarization vector. Hence, we can write $f_{\mu\nu} = k_\mu\,a_\nu - k_\nu\,a_\mu$ on Σ. From equation (4) it follows that $f^{\mu\nu} k_\nu = 0$, and contracting (5) with $\eta^{\lambda\rho} k_\rho$ yields $f_{\mu\nu}\eta^{\alpha\beta} k_\alpha k_\beta = 0$. Therefore, such modes propagate along standard null geodesics in Minkowski spacetime.

where we have chosen an Euclidean coordinate system in which $\gamma_{\mu\nu}$ reduces to $\eta_{\mu\nu}$. In terms of this tensor the effective geometry (8) can be re-written as[c]

$$g^{\mu\nu} = \left(L_F + \frac{L\, L_{FF}}{L_F} \right) \eta^{\mu\nu} + \frac{L_{FF}}{L_F} T^{\mu\nu}. \tag{11}$$

We remark that, once the modified geometry along which the photon propagates depends upon the energy-momentum tensor distribution of the background electromagnetic field, it is tempting to search for an analogy with the corresponding behavior of photons in a gravitational field. We will return to this question later on.

Therefore, the field discontinuities propagate along null geodesics in an effective geometry which depends on the field energy distribution. Let us point out that, as it is explicitly shown from the above equation, the stress-energy distribution of the field is the true responsible for the deviation of the geometry, as felt by photons, from its Minkowskian form.[d]

In order to show that the photon path is actually a geodesic curve, it is necessary to know the inverse $g^{\mu\nu}$ of the effective metric $g_{\nu\lambda}$, defined by

$$g^{\mu\nu}\, g_{\nu\lambda} = \delta^{\mu}_{\lambda}. \tag{12}$$

This calculation is simplified if we take into account the well known properties:

$$F^{*}_{\mu\nu}\, F^{\nu\lambda} = -\,\frac{1}{4}\, G\, \delta^{\lambda}_{\mu}, \quad F^{*}_{\mu\lambda}\, F^{*\,\lambda\nu} - F_{\mu\lambda}\, F^{\lambda\nu} = \frac{1}{2}\, F\, \delta^{\nu}_{\mu}. \tag{13}$$

Thus the covariant form of the metric can be written in the form:

$$g_{\mu\nu} = a\, \eta_{\mu\nu} + b\, T_{\mu\nu}, \tag{14}$$

in which a and b are given in terms of the Lagrangian and derivatives by:

$$\begin{aligned} a &= -\,b \left(\frac{L_F^2}{L_{FF}} + L + \frac{1}{2} T \right), \\ b &= 16\, \frac{L_{FF}}{L_F} \left[\left(F^2 + G^2 \right) L_{FF}^2 - 16\, \left(L_F + F\, L_{FF} \right)^2 \right]^{-1}, \end{aligned} \tag{15}$$

where $T = T^{\alpha}_{\alpha}$ is the trace of the energy-momentum tensor.

3. The Effective Null Geodesics

The geometrical relevance of the effective geometry goes beyond its immediate definition. Indeed, as follows it will be shown that the integral curves of the vector k_ν (*i.e.*, the photons trajectories) are in fact geodesics. In order to achieve this result it will be required an underlying Riemannian structure for the manifold associated with the effective geometry. In other words this implies a set of Levi-Civita

[c]For simplicity, we will denote the effective metric as $g^{\mu\nu}$ instead of $g^{\mu\nu}_{\rm eff}$ from now on.
[d]For $T_{\mu\nu} = 0$, the conformal modification in (11) clearly leaves the photon paths unchanged.

connection coefficients $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$, by means of which there exists a covariant differential operator ∇_λ (the *covariant derivative*) such that

$$\nabla_\lambda g^{\mu\nu} \equiv g^{\mu\nu}{}_{;\,\lambda} \equiv g^{\mu\nu}{}_{,\,\lambda} + \Gamma^{\mu}{}_{\sigma\lambda} g^{\sigma\nu} + \Gamma^{\nu}{}_{\sigma\lambda} g^{\sigma\mu} = 0. \tag{16}$$

From (16) it follows that the effective connection coefficients are completely determined from the effective geometry by the usual Christoffel formula.

Contracting (16) with $k_\mu k_\nu$ results

$$k_\mu k_\nu g^{\mu\nu}{}_{,\,\lambda} = -2 k_\mu k_\nu \Gamma^{\mu}{}_{\sigma\lambda} g^{\sigma\nu}. \tag{17}$$

Differentiating (29) we have

$$2 k_{\mu,\,\lambda} k_\nu g^{\mu\nu} + k_\mu k_\nu g^{\mu\nu}{}_{,\,\lambda} = 0. \tag{18}$$

Inserting (17) for the last term on the left hand side of (18) we obtain

$$g^{\mu\nu} k_{\mu;\,\lambda} k_\nu \equiv g^{\mu\nu} \left(k_{\mu,\,\lambda} - \Gamma^{\sigma}{}_{\mu\lambda} k_\sigma\right) k_\nu = 0. \tag{19}$$

As the propagation vector $k_\mu = \Sigma_{,\,\mu}$ is an exact gradient one can write $k_{\mu;\,\lambda} = k_{\lambda;\,\mu}$. With this identity and defining $k^\mu \doteq g^{\mu\nu} k_\nu$ equation (19) reads $k_{\mu;\,\lambda} k^\lambda = 0$, which states that k_μ is a geodesic vector. By remembering it is also a null vector (with respect to the effective geometry $g^{\mu\nu}$), it follows that its integral curves are therefore null geodesics.

3.1. *Two parameter Lagrangians*

In this section we will go one step further and deal with the general case in which the effective action depends upon both invariants, that is $L = L(F, G)$. The equations of motion are

$$\partial_\nu \left(L_F F^{\mu\nu} + L_G F^{*\,\mu\nu}\right) = 0. \tag{20}$$

Our aim is to examine the propagation of the discontinuities in such case. Following the same procedure as presented in the previous section one gets

$$\left[L_F\, f^{\mu\nu} + 2A\, F^{\mu\nu} + 2B\, F^{*\,\mu\nu}\right] k_\nu = 0, \tag{21}$$

and contracting this expression with $F^{\alpha}{}_{\mu} k_\alpha$ and with $F^{*\,\alpha}{}_{\mu} k_\alpha$, respectively, yields

$$\left[\xi\, L_F + \frac{1}{2}\, B\, G\right] \eta^{\mu\nu}\, k_\mu\, k_\nu - 2A F^{\nu}{}_{\alpha}\, F^{\alpha\mu} k_\nu k_\mu = 0 \tag{22}$$

$$\left[\zeta\, L_F - B\, F + \frac{1}{2}\, A\, G\right] \eta^{\mu\nu}\, k_\mu\, k_\nu - 2B F^{\nu}{}_{\alpha}\, F^{\alpha\mu} k_\nu k_\mu = 0. \tag{23}$$

In these expressions we have set $A \doteq 2\,(\xi\, L_{FF} + \zeta\, L_{FG})$, $B \doteq 2\,(\xi\, L_{FG} + \zeta\, L_{GG})$, and $\zeta \doteq F^{\alpha\beta}\, f^{*}_{\alpha\beta}$. In order to simplify our equations it is worth defining the quantity $\Omega \doteq \zeta/\xi$. From equations (22) and (23) it follows

$$\Omega^2\, \Omega_1 + \Omega\, \Omega_2 + \Omega_3 = 0, \tag{24}$$

with the quantities $\Omega_i,\ i = 1, 2, 3$ given by

$$\Omega_1 = -L_F L_{FG} + 2F L_{FG} L_{GG} + G(L_{GG}^2 - L_{FG}^2), \tag{25}$$

$$\Omega_2 = (L_F + 2G L_{FG})(L_{GG} - L_{FF}) + 2F(L_{FF} L_{GG} + L_{FG}^2), \tag{26}$$

$$\Omega_3 = L_F L_{FG} + 2F L_{FF} L_{FG} + G(L_{FG}^2 - L_{FF}^2). \tag{27}$$

The quantity Ω is then given by the algebraic expression

$$\Omega = \frac{-\Omega_2 \pm \sqrt{\Delta}}{2\Omega_1}, \tag{28}$$

where $\Delta \doteq (\Omega_2)^2 - 4\Omega_1\Omega_3$. Thus, in the general case we are concerned here, the photon path is kinematically described by

$$g^{\mu\nu} k_\mu k_\nu = 0, \tag{29}$$

where the effective metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = L_F \eta^{\mu\nu} - 4\left[(L_{FF} + \Omega L_{FG}) F^\mu{}_\lambda F^{\lambda\nu} + (L_{FG} + \Omega L_{GG}) F^\mu{}_\lambda F^{*\,\lambda\nu}\right]. \tag{30}$$

When the Lagrangian does not depend on the invariant G, expression (30) reduces to the form (8).

From the general expression of the energy-momentum tensor for an electromagnetic theory $L = L(F, G)$ we have

$$T_{\mu\nu} = -4L_F\, F_\mu{}^\alpha\, F_{\alpha\nu} - (L - G\, L_G)\, \eta_{\mu\nu}. \tag{31}$$

The scale anomaly is given by the trace

$$T = 4\,(-L + F\, L_F + G\, L_G)\,. \tag{32}$$

We can then re-write the effective geometry in a more appealing form in terms of the energy momentum tensor, that is,

$$g^{\mu\nu} = \mathcal{M}\, \eta^{\mu\nu} + \mathcal{N}\, T^{\mu\nu}, \tag{33}$$

where the functions $\mathcal{M}$ and $\mathcal{N}$ are given by

$$\mathcal{M} = L_F + G\,(L_{FG} + \Omega L_{GG}) + \frac{1}{L_F}\,(L_{FF} + \Omega L_{FG})\,(L - G L_G)\,, \tag{34}$$

$$\mathcal{N} = \frac{1}{L_F}\,(L_{FF} + \Omega L_{FG})\,. \tag{35}$$

As a consequence of this, the Minkowskian norm of the propagation vector k_μ reads

$$\eta^{\mu\nu} k_\mu\, k_\nu = -\frac{\mathcal{N}}{\mathcal{M}} T^{\mu\nu} k_\mu k_\nu. \tag{36}$$

3.2. *Wave propagation in nonlinear dielectric media*

It is possible to describe the wave propagation, governed by Maxwell electrodynamics, inside a dielectric, in terms of a modification of the underlying spacetime geometry using the framework developed above. The electromagnetic field is represented by two antisymmetric tensors, the electromagnetic field $F_{\mu\nu}$ and the polarization $P_{\mu\nu}$. These tensors are decomposed, in the standard way, into their corresponding electric and magnetic parts as seen by an observer which moves with velocity v_μ. We can write:

$$F_{\mu\nu} = E_\mu\, v_\nu - E_\nu\, v_\mu + \eta^{\rho\sigma}{}_{\mu\nu}\, v_\rho\, H_\sigma, \quad P_{\mu\nu} = D_\mu\, v_\nu - D_\nu\, v_\mu + \eta^{\rho\sigma}{}_{\mu\nu}\, v_\rho\, B_\sigma. \tag{37}$$

Following Hadamard, we consider the discontinuities on the fields as given by $[\bigtriangledown_\lambda\, E_\mu]_\Sigma = k_\lambda\, e_\mu$, $[\bigtriangledown_\lambda\, D_\mu]_\Sigma = k_\lambda\, d_\mu$, $[\bigtriangledown_\lambda\, H_\mu]_\Sigma = k_\lambda\, h_\mu$, $[\bigtriangledown_\lambda\, B_\mu]_\Sigma = k_\lambda\, b_\mu$. For the simplest linear case, in which we have $D_\alpha = \epsilon\, E_\alpha$, $B_\alpha = \frac{H_\alpha}{\mu}$, it follows that $d_\alpha = \epsilon\, e_\alpha$, $b_\alpha = h_\alpha/\mu$. After a straightforward calculation one obtains

$$k_\mu\, k_\nu\, [\gamma^{\mu\nu} + (\epsilon\,\mu - 1) v^\mu\, v^\nu] = 0. \tag{38}$$

Let us generalize this situation for the nonlinear case. Maxwell equations are

$$\partial^\nu\, F^*_{\mu\nu} = 0, \quad \partial^\nu\, P_{\mu\nu} = 0. \tag{39}$$

For electrostatic fields inside isotropic dielectrics $P^{\mu\nu}$ and $F^{\mu\nu}$ are related by $P_{\mu\nu} = \epsilon(E) F_{\mu\nu}$, where ϵ is the electric susceptibility. In the general case, for $\epsilon = \epsilon\,(E)$ we simplify our calculation if we note that we can relate the equation of wave propagation to the previous analysis on vacuum polarization by means of the identification $L_F \longrightarrow \epsilon$, which implies $L_{FF} \longrightarrow -\,\epsilon'/(4E)$, where $\epsilon' \equiv d\,\epsilon/d\,E$. Therefore, the simple class of effective Lagrangians may be used as a convenient description of Maxwell theory inside isotropic nonlinear dielectric media; conversely, results obtained in this context can as well be similarly restated in the former one.

In a nonlinear dielectric medium the polarization induced by an external electric field is described by expressing the scalar function ϵ as a power series in terms of the field strength E:

$$\epsilon = \chi_1 + \chi_2\, E + \chi_3\, E^2 + \chi_4\, E^3 + \dots\,, \tag{40}$$

where the constants χ_n are known as the n-order nonlinear optical susceptibility. Note that we are using the standard convention which relates χ_n with the expansion of the polarization vector. For this case the effective geometry is given by

$$g^{\mu\nu} = \epsilon\, \eta^{\mu\nu} + \frac{\epsilon'}{E} F^\mu{}_\alpha F^{\alpha\nu}. \tag{41}$$

It can also be re-written in the form

$$g^{\mu\nu} = \epsilon\, \eta^{\mu\nu} - \frac{\epsilon'}{E}\left(E^\mu\, E^\nu - E^2\, \delta^\mu_t\, \delta^\nu_t\right), \tag{42}$$

where $E^2 \equiv -\, E_\alpha\, E^\alpha > 0$. In other words,

$$g^{tt} = \epsilon + \epsilon' E \quad g^{ij} = -\,\epsilon\, \delta^{ij} - \frac{\epsilon'}{E}\, E^i\, E^j. \tag{43}$$

This shows that the discontinuities of the electromagnetic field in a nonlinear dielectric medium propagate along null cones of an effective geometry which depends on the characteristics of the medium given by Eq. (41). It seems worth investigating under what conditions of the ϵ dependence on E a kind of horizon barrier should appear for the photon in a dielectric. We will return to this problem elsewhere.

4. Moving Dielectrics

For the case of moving dielectric medium we proceed in an analogous way as in the previous section. We set

$$[P_{\mu\nu}]_\Sigma = 0, \quad [\partial_\lambda\, P_{\mu\nu}]_\Sigma = p_{\mu\nu} k_\lambda. \tag{44}$$

It is convenient to project these tensors in the framework of a real observer endowed with normalized four-velocity v^μ, thus defining the corresponding electric and magnetic vectors in the 3-dimensional rest-space of the observer v^μ :

$$F_{\mu\nu} = E_\mu\, v_\nu - E_\nu\, v_\mu + \eta_{\mu\nu}^{\rho\sigma}\, v_\rho\, B_\sigma; \quad P_{\mu\nu} = D_\mu\, v_\nu - D_\nu\, v_\mu + \eta_{\mu\nu}^{\rho\sigma}\, v_\rho\, H_\sigma.$$

Accordingly we decompose the discontinuity tensors into corresponding electric and magnetic parts: $f_{\mu\nu} \leftrightarrow (e_\mu, b_\mu)$, $p_{\mu\nu} \leftrightarrow (d_\mu, h_\mu)$. The equations of motion are:

$$\partial_\nu\, P^{\mu\nu} = 0, \quad \partial^\nu\, F^*_{\mu\nu} = 0. \tag{45}$$

Following the definitions and procedure presented above, one gets from the discontinuity of the equation of motion Eq. (45):

$$\left(b_\mu\, v_\nu - b_\nu\, v_\mu - \eta_{\mu\nu}^{\rho\sigma}\, v_\rho\, e_\sigma\right)\, k_\nu = 0, \;\; \left(d_\mu\, v_\nu - d_\nu\, v_\mu - \eta_{\mu\nu}^{\rho\sigma}\, v_\rho\, h_\sigma\right)\, k_\nu = 0. \tag{46}$$

In the present article we shall focus our analysis on the case in which the polarization tensor is such that $D_\alpha = \epsilon\, E_\alpha$ and $B_\alpha = \mu\, H_\alpha$. Besides, we take the dielectric permittivity to be a real function of the electric field, that is $\epsilon = \epsilon(E)$. Multiplying the equation of the discontinuity by v^μ we obtain that $b_\mu\, k^\mu = 0$ and $d_\mu\, k^\mu = 0$. Thus, it follows that $h_\mu\, k^\mu = 0$ and

$$e_\mu\, k^\mu = \frac{\epsilon^{'}}{\epsilon} \frac{(E_\mu\, k^\mu)^2}{E},$$

in which $\epsilon^{'} \equiv d\epsilon/dE$. From the discontinuity equation we get

$$h_\mu = \frac{1}{\mu\,(k^\alpha\, v_\alpha)}\, \eta_{\mu\nu\rho\sigma}\, v^\rho\, e^\sigma\, k^\nu. \tag{47}$$

Substituting this expression into the other discontinuity equation and multiplying by E^μ we get after some algebraic manipulations

$$\left[\eta^{\mu\nu} + v^\mu\, v^\nu\, \left(\mu\, \epsilon - 1 + \mu\, \epsilon^{'}\, E\right) - \frac{\epsilon^{'}}{\epsilon\, E} E^\mu\, E^\nu\right] k_\mu\, k_\nu = 0. \tag{48}$$

It then follows that the photon path is kinematically described by Eq.(29) where the effective metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \eta^{\mu\nu} + v^\mu\, v^\nu \left(\mu\,\epsilon - 1 + \mu\,\epsilon^{'}\, E\right) - \frac{\epsilon^{'}\, E}{\epsilon} \ell^\mu\, \ell^\nu. \tag{49}$$

Here ℓ^μ is the unitary vector in the direction of the electric field. In the particular case in which ϵ is a constant, this formula goes into the reduced Gordon geometry

$$g^{\mu\nu}_{Gordon} = \eta^{\mu\nu} + v^\mu\, v^\nu\, (\mu\,\epsilon - 1)\,. \tag{50}$$

The inverse metric $g_{\mu\nu}$, defined by $g^{\mu\nu}\, g_{\nu\alpha} = \delta^\mu_\alpha$, is

$$g_{\mu\nu} = \eta_{\mu\nu} - v_\mu\, v_\nu \left(1 - \frac{1}{\mu\,\epsilon(1+\xi)}\right) + \frac{\xi}{1+\xi}\, \ell_\mu\, \ell_\nu, \tag{51}$$

where we have set $\xi \equiv \frac{\epsilon^{'}\, E}{\epsilon}$. We note that, once the wave vector k_α is a gradient, the photon path is a true geodesic in the effective geometry. We obtain then the remarkable result that the discontinuities of the electromagnetic field in a nonlinear electrodynamics propagate along null geodesics of an effective geometry which depends on the properties of the background field.

The velocity of the photon $v_{ph} = k_\mu\, v^\mu/|k|$, where $|k| \equiv (\eta^{\mu\nu} - v^\mu\, v^\nu) k_\mu k_\nu$, is

$$v_{ph} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\frac{1 + \xi cos^2\theta}{1+\xi}},$$

in which θ is the angle between the direction of the electric field and the propagation of the photon. Note that in the limit case in which ξ vanishes, the photon velocity coincides with the square-root of the determinant of the effective metric. Indeed, for a geometry given by $g_{\mu\nu} = \eta_{\mu\nu} + a_{\mu\nu}$ where $a_{\mu\nu}$ is symmetric, we have

$$\begin{aligned} det\, g_{\mu\nu} = {} & 1 + a + \frac{1}{2}\, a^2 + \frac{1}{6}\, a^3 + \frac{1}{24}\, a^4 - \frac{1}{2}(\hat{a}\hat{a}) + \frac{1}{3}\,(\hat{a}\hat{a}\hat{a}) \\ & - \frac{1}{4}\,(\hat{a}\hat{a}\hat{a}\hat{a}) - \frac{1}{2}\, a\,(\hat{a}\hat{a}) + \frac{1}{3}\, a\,(\hat{a}\hat{a}\hat{a}) - \frac{1}{4}\, a^2\,(\hat{a}\hat{a}) + \frac{1}{8}\,(\hat{a}\hat{a})^2, \end{aligned} \tag{52}$$

where $a \equiv a^\mu_\mu$, $\hat{a}\hat{a} \equiv a^\mu_\nu\, a^\nu_\mu$, $\hat{a}\hat{a}\hat{a} \equiv a^\nu_\mu\, a^\mu_\alpha\, a^\alpha_\nu$, $\hat{a}\hat{a}\hat{a}\hat{a} \equiv a^\nu_\mu\, a^\mu_\alpha\, a^\alpha_\beta\, a^\beta_\nu$.

Using this property it follows that the determinant of the effective metric is

$$det\, g_{\mu\nu} = \frac{1}{\mu\epsilon\,(1+\xi)^2.} \tag{53}$$

In the case ϵ does not depend on the electric field, the photon velocity can be written in terms of the determinant of the effective metric $v_{ph} = \sqrt{g}$.

The developed method can be applied to display properties of the photon propagation in an arbitrary dielectric medium. It can be shown that the effective metric can mimic some properties of the geometry discovered by Gödel in Einstein General Relativity, where closed paths in spacetime occur. As a consequence, the circular orbits for the photons (although unstable) turn out to be possible.[1]

5. Preliminary Synthesis

The propagation of discontinuities of electromagnetic field in a nonlinear regime (as it occurs, for instance, in dielectrics or in modified QED vacua) can be described in terms of an effective modification of the geometry of spacetime. Such interpretation is an immediate consequence of the analysis we presented here. This description also allows us to recognize a striking analogy between photon propagation in nonlinear electrodynamics and its behavior in an external gravitational field. For in both cases the geometry is modified by a nonlinear process. It is clear that such analogy can not be pushed very far, since in the gravitational case the modified geometry is observed by any kind of matter and energy (including gravitational energy — at least in the GR) and in the electromagnetic case this modified geometry is observed only by the nonlinear photons. This analogy certainly deserves further examination, since it may provide for the existence of an electromagnetic analogue of the gravitational black hole.

It seems worth to make some comments in order to avoid possible misunderstandings. The analysis that Gödel carried out in GR only makes sense for the nonlinear photons. These propagate following geodesics of the effective metric. All other particles, interactions and observers propagate in the Minkowski background. A class of syncronized inertial observers on a hypersurface $x^0 = constant$, that contains the closed curve, will see this photon path as a closed spacelike curve.

It has been known from more than half a century that gravitational processes allow the existence of closed paths in spacetime. This led to the belief that this strange situation occurs uniquely under the effect of gravity. As we have shown this is not the case: photons can follow closed curves due to electromagnetic forces in a non-linear regime. This new property depends crucially on the non-linearity of the electromagnetic processes and it does not exist in Maxwell's theory.

To close this section we emphasize that the existence of closed curves is not an exclusive property of the gravitational interaction. The existence of such curves in both gravitational and electromagnetic processes asks for a deep review of the causal structure as displayed by the geodesics of the photons.

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