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Constructing Dirac linear fermions in terms of non-linear Heisenberg spinors

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Abstract – We show that the massive (or massless) neutrinos can be described as special states of Heisenberg nonlinear spinors. As a by-product of this decomposition a particularly attractive consequence appears: the possibility of relating the existence of only three species of mass-less neutrinos to such internal non-linear structure. At the same time it allows the possibility that neutrino oscillation can occur even for massless neutrinos.

There are evidences that neutrino changes from one flavor to another as observed for instance in neutrino oscillations found by the Super-Kamiokande Collaboration [1,2]. This mix is understood as an evidence that the neutrino has a small mass [3]. This has important consequences not only in local laboratory experiments but also in astrophysics and even in cosmology. In a closely related path, the possibility that not only left-handed but also right-handed neutrinos exist has recently attracted interest, receiving a new treatment in a very imaginative example presented in [4] dealing with the possibility of neutrino superfluidity. The main idea requires the existence of an interaction between neutrinos that in the case of small energy and momentum can be described as a sort of Fermi process involving terms like $(\overline{\nu}\nu)(\overline{\nu}\nu)$. If the ν's are the same, this interaction is nothing but an old theory of Heisenberg concerning self-interacting fermions [5].

Recent experiments [6] strongly support the idea that there are only three neutrino flavours. Based on this and on the possibility of mixing neutrino species, it has been argued [3] that neutrino flavours ν_α are combinations of mass eigenstates ν_i of mass m_i through a unitary $N \times N$ matrix $U_{\alpha\, i}$.

It would be interesting if we could describe all these properties as consequences of the existence of a common root for the neutrino species, *e.g.*, if they are particular realizations of a unique structure. In this paper we will develop a model of such idea and work out a unified description of the three species of neutrinos by showing that they can be considered as having a common origin on a more fundamental nonlinear structure. Actually such property is not exclusive for neutrinos but instead is typical for any Dirac fermion (*e.g.*, quark, electron). However as we shall see, the decomposition of the Dirac fermion in terms of non-linear structure contains three parameters (associated to the Heisenberg self-interaction constant) that separate different classes of Dirac spinors and three elements for each class that could be associated to three types of particles in each class. This form of decomposition may appear as if we were inverting the common procedure and treating the simple linear case of Dirac spinor as a particular state of a more involved self-interacting nonlinear structure. This goes in the same direction as some modern treatments in which linearity is understood as a realization of a subjacent nonlinear structure. In this vein we will examine the hypothesis that neutrinos are special states of nonlinear Heisenberg spinors.

The argument is based on two fundamental equations: the linear Dirac equation of motion

$$i\gamma^\mu \partial_\mu \Psi^D - M\,\Psi^D = 0 \tag{1}$$

and the non linear Heisenberg theory[1]

$$i\gamma^\mu \partial_\mu \Psi^H - 2s\,(A + iB\gamma^5)\,\Psi^H = 0 \tag{2}$$

in which the constant s has the dimension of (lenght)^2 and the quantities A and B are given in terms of the Heisenberg spinor Ψ^H as

$$A \equiv \overline{\Psi}^H \Psi^H \tag{3}$$

[1]This equation represents a self-interacting spinor field driven by a Lagrangian of the form typical of Fermi processes, *e.g.*, $L_{int} = sJ_\mu J^\mu$.

and

$$B \equiv i\overline{\Psi}^H \gamma^5 \Psi^H \quad (4)$$

The Heisenberg spinor Ψ^H can be depicted as a line making an angle of 45 degrees with each axis representing Ψ_L and Ψ_R in the two-dimensional plane π generated by left-hand and right-hand spinors as follows from the identity:

$$\Psi^H = \Psi^H_L + \Psi^H_R = \frac{1}{2}(1+\gamma^5)\Psi^H + \frac{1}{2}(1-\gamma^5)\Psi^H. \quad (5)$$

The main outcome of the present paper is the proof of the statement that a massive or massless neutrino that satisfies Dirac equation (1) can be described as a deformation of the Heisenberg spinor in the π plane. We are, indeed, claiming that it is possible to write the Dirac spinor as a deformation of Ψ^H in the plane π given by

$$\Psi^D = e^F\, \Psi^H_L + e^G\, \Psi^H_R, \quad (6)$$

or, in other words, that the left- and the right-handed Dirac spinors are given by $\Psi^D_L = e^{\mathrm{F}}\, \Psi^H_L$ and $\Psi^D_R = e^G\, \Psi^H_R$. What are the properties of F and G in order that Ψ^D satisfies eq. (1)?

The Inomata solution of Heisenberg dynamics. – In [7] a particular class of solutions of Heisenberg equation was set out. The interest on this class rests on the fact that it directly allows one to deal with the derivatives of the spinor field, consequently allowing to obtain derivatives of any associated function of the spinor. Let us briefly present this class of spinors. The analysis starts by the recognition that it is possible to construct a sub-class of solution of Heisenberg dynamics by imposing a restrictive condition given by

$$\partial_\mu \Psi = \left(a\, J_\mu + b\, I_\mu \gamma^5\right)\, \Psi, \quad (7)$$

where a and b are complex numbers of dimensionality $(\text{lenght})^2$. This is a generalization, for the non-linear case, of a similar condition in the linear case provided by plane waves, that is, $\partial_\mu \Psi = ik_\mu\, \Psi$.

A Ψ that satisfies condition (7) will be called an Inomata spinor. It is immediate to prove that if Ψ satisfies this condition it satisfies automatically Heisenberg equation of motion if a and b are such that $2s = i\,(a-b)$.

This is a rather strong condition that deals with simple derivatives instead of the scalar structure obtained by the contraction with γ_μ typical of Dirac or even for the Heisenberg operators that appear in both equations (1) and (2). Prior to anything one has to examine the compatibility of such condition which concerns all quantities that can be constructed with such spinors. It is a remarkable result that, in order that the restrictive condition eq. (7) be integrable, constants a and b must satisfy a unique constraint given by $\mathrm{Re}(a) - \mathrm{Re}(b) = 0$.

Indeed, we have

$$[\partial_\mu, \partial_\nu]\, \Psi = \left(a\, \partial_{[\mu}\, J_{\nu]} + b\, \partial_{[\mu}\, I_{\nu]}\, \gamma^5\right)\, \Psi.$$

Now, the derivative of the currents yields

$$\partial_\mu J_\nu - \partial_\nu J_\mu = (a+\overline{a})[J_\mu,\, J_\nu] + (b+\overline{b})\, [I_\mu,\, I_\nu]$$

and

$$\partial_\mu I_\nu - \partial_\nu I_\mu = (a+\overline{a}-b-\overline{b})[J_\mu\, I_\nu - I_\mu\, J_\nu].$$

Thus the condition of integrability is given by

$$\mathrm{Re}(a) = \mathrm{Re}(b). \quad (8)$$

It is a rather long and tedious work to show that any combination X constructed with Ψ and for all elements of the Clifford algebra, the compatibility condition $[\partial_\mu, \partial_\nu]X = 0$ is automatically fulfilled once this unique condition (8) is satisfied.

Let us now turn to some remarkable properties of I-spinors.

Lemma. The current four-vector J^μ is irrotational. The same is valid for the axial-current I^μ.

Proof. The proof that the vector J^μ is the gradient of a certain scalar quantity is a simple direct consequence of its definition in terms of H-spinors. However there is a further property that is worth of mention: this scalar is nothing but the norm J^2 of the current. Indeed, using equation (7), we have

$$\partial_\mu J_\nu = (a+\overline{a}) J_\mu J_\nu + (b+\overline{b}) I_\mu I_\nu. \quad (9)$$

This expression shows that the derivative of the four-vector current is symmetric. Multiplying eq. (9) by J^μ it follows then

$$J_\mu = \partial_\mu S, \quad (10)$$

in which the scalar S is written in terms of the norm $J^2 \equiv J_\mu J^\mu$:

$$S = \frac{1}{a+\overline{a}} \ln\sqrt{J^2}. \quad (11)$$

Note that $S = \text{const}$ defines a hypersurface in space-time such that the current J_μ is not only geodesic but orthogonal to S. It follows that

$$\partial_\mu S\, \partial_\nu S\, \eta^{\mu\nu} = e^{2(a+\overline{a})S},$$

or, defining the conformal metric

$$g^c_{\mu\nu} \equiv e^{2(a+\overline{a})\, S}\, \eta_{\mu\nu},$$

we write

$$\partial_\mu S\, \partial_\nu S\, g^{\mu\nu}_c = 1, \quad (12)$$

showing that S is an eikonal in the associated conformal space.

Lemma. The two four-vectors J_μ and I_μ constitute a basis for vectors constructed by the derivative ∂_μ operating on functionals of Ψ.

Proof. It is enough to show that this assertion is true for the scalars A and B. Indeed, we have

$$\partial_\mu\, A = (a+\overline{a})\, A\, J_\mu + (b-\overline{b})\, iB\, I_\mu \quad (13)$$

and

$$\partial_\mu B = (a+\bar{a})\, B\, J_\mu + (b-\bar{b})\, iAI_\mu. \qquad (14)$$

It then follows that the vector I_μ is a gradient too. Indeed,

$$\partial_\mu I_\nu = (a+\bar{a}) J_\mu I_\nu + (b+\bar{b}) J_\nu I_\mu, \qquad (15)$$

$$I_\mu = \partial_\mu R, \qquad (16)$$

in which the scalar R is

$$R = \frac{1}{b-\bar{b}} \ln\left(\frac{A-iB}{\sqrt{J^2}}\right). \qquad (17)$$

From Heisenberg to Dirac: How elementar is the neutrino? – In this section we will describe an unexpected result of the Inomata class $\mathcal{IC}$ which states that for any spinor of $\mathcal{IC}$ it is possible to construct another spinor which satisfies the linear Dirac equation, the neutrino, for instance. In other words, we claim that a spinor that satisfies the linear Dirac equation may be constructed in terms of a nonlinear structure. Let us prove this statement. We start by defining the deformation in the plane π_H characterized by the left- and right-handed Heisenberg spinors by setting for the left- and the right-handed Dirac spinor the expressions

$$\Psi_L^D = e^F\, \Psi_L^H, \qquad (18)$$

$$\Psi_R^D = e^G\, \Psi_R^H. \qquad (19)$$

What are the properties of F and G in order that Ψ^D satisfies Dirac equation? In order to answer this question, we have to make some additional calculations. From eq. (7) we obtain

$$\partial_\mu \Psi_L^H = (a\, J_\mu + b\, I_\mu)\, \Psi_L^H, \qquad (20)$$

$$\partial_\mu \Psi_R^H = (a\, J_\mu - b\, I_\mu)\, \Psi_R^H. \qquad (21)$$

Now comes the miracle that permits the accomplishment of our procedure, which is the fact that the two vectors J_μ and I_μ can be written as gradients of nonlinear expressions under the form

$$\begin{aligned} J_\mu &= \partial_\mu S, \\ I_\mu &= \partial_\mu R, \end{aligned} \qquad (22)$$

where S and R are given in eqs. (11) and (17). From these equations it follows that

$$\partial_\mu \Psi_L^D = \left(\frac{\partial F}{\partial S}\, J_\mu + \frac{\partial F}{\partial R}\, I_\mu\right) \Psi_L^D + (a\, J_\mu + b\, I_\mu)\, \Psi_L^D \qquad (23)$$

and

$$\partial_\mu \Psi_R^D = \left(\frac{\partial G}{\partial S}\, J_\mu + \frac{\partial G}{\partial R}\, I_\mu\right) \Psi_R^D + (a\, J_\mu - b\, I_\mu)\, \Psi_R^D. \qquad (24)$$

Multiplying these expressions by $i\,\gamma^\mu$ it follows that Ψ^D satisfies Dirac equation, if F and G are given by

$$F = -\frac{1}{2}(b-\bar{b})\, R + \left(2is - \frac{1}{2}(b-\bar{b})\right) S + \frac{iM}{a+\bar{a}}\, e^{-(a+\bar{a})S}, \qquad (25)$$

$$G = \frac{1}{2}(b-\bar{b})\, R + \left(2is - \frac{1}{2}(b-\bar{b})\right) S + \frac{iM}{a+\bar{a}}\, e^{-(a+\bar{a})S}. \qquad (26)$$

To arrive at this result it is convenient to use the formulas provided by Pauli-Kofink identities (see the appendix) to obtain

$$\begin{aligned} J_\mu\, \gamma^\mu\, \Psi_L &= (A-iB)\, \Psi_R, \\ I_\mu\, \gamma^\mu\, \Psi_L &= -(A-iB)\, \Psi_R, \\ I_\mu\, \gamma^\mu\, \Psi_R &= (A+iB)\, \Psi_L, \\ J_\mu\, \gamma^\mu\, \Psi_R &= (A+iB)\, \Psi_L, \end{aligned} \qquad (27)$$

where

$$A + iB = \frac{J^2}{A-iB}.$$

Thus, the linear Dirac field can be written in terms of the non-linear Heisenberg field by

$$\Psi_L^D = \sqrt{\frac{J}{A-iB}}\, \exp\left(\frac{iM}{(a+\bar{a})\, J} + \left(2is - \frac{1}{2}(b-\bar{b})\right) S\right) \Psi_L^H, \qquad (28)$$

$$\Psi_R^D = \sqrt{\frac{A-iB}{J}}\, \exp\left(\frac{iM}{(a+\bar{a}) J} + \left(2is - \frac{1}{2}(b-\bar{b})\right) S\right) \Psi_R^H, \qquad (29)$$

where $J \equiv J^2$. Using expression (11) we can simplify these expressions, then we can write

$$\exp\left(2is - \frac{1}{2}(b-\bar{b})\right) S = J^{2\sigma},$$

where we have defined

$$\sigma \equiv \frac{is - \frac{1}{4}(b-\bar{b})}{a+\bar{a}} = -\frac{i}{4}\, \frac{\mathrm{Im}(a)}{\mathrm{Re}(a)}.$$

Then, finally, for the Dirac spinor

$$\Psi^D = \exp\frac{iM}{(a+\bar{a})\, J}\, J^{2\sigma}\left(\sqrt{\frac{J}{A-iB}}\Psi_L^H + \sqrt{\frac{A-iB}{J}}\, \Psi_R^H\right), \qquad (30)$$

or, for the mass-less neutrino

$$\Psi^D = J^{2\sigma}\, \left(\sqrt{\frac{J}{A-iB}}\Psi_L^H + \sqrt{\frac{A-iB}{J}}\, \Psi_R^H\right). \qquad (31)$$

This finally proves the following

Lemma: A free linear massive (or massless) Dirac field can be represented as a combination of Inomata spinors satisfying the non-linear Heisenberg equation.

We must analyze carefully the domain of parameters a and b once the potentials S and R become singular in the imaginary axis and in the real axis, respectively. Thus we can distinguish different domains in the space of these two parameters. We set $a = a_0\, e^{i\varphi}$ and $b = b_0\, e^{i\theta}$. Then, the constraints on these previously presented parameters, which allow for the existence of the Inomata solution, are written under the form:

$$\frac{\cos\varphi}{\cos\theta} > 0, \tag{32}$$

$$\cos\varphi\,(\tan\varphi - \tan\theta) < 0, \tag{33}$$

once the Heisenberg constant s is positive. Let us name the following sectors: W_1 for $0 < \varphi < \frac{\pi}{2}$; W_2 for $\frac{\pi}{2} < \varphi < \pi$; W_3 for $\pi < \varphi < \frac{3\pi}{2}$, and W_4 for $\frac{3\pi}{2} < \varphi < 2\pi$. In an analogous way we define Z_1, Z_2, Z_3 and Z_4 for similar sectors of θ. We distinguish then six domains:

$$\Omega_1 \equiv W_1 \otimes Z_1,$$

$$\Omega_2 \equiv W_4 \otimes Z_1,$$

$$\Omega_3 \equiv W_4 \otimes Z_4,$$

$$\Omega_4 \equiv W_2 \otimes Z_2,$$

$$\Omega_5 \equiv W_3 \otimes Z_2,$$

$$\Omega_6 \equiv W_3 \otimes Z_3.$$

The missing domains $W_1 \otimes Z_4$ and $W_2 \otimes Z_3$ are forbidden because they violate constraint (33). Thus, for the massless case, eq. (31) shows that different choices of the parameters a and b for a given value of constant s yield different spinor configurations Ψ^D. This allows us to write

$$\Psi^D = \sum_{\Omega_i} c_i\, \Gamma^{i,s}, \tag{34}$$

where $\Gamma^{i,s}$ is defined by the rhs of eq. (31) and we have to sum over all possible independent domains. Note furthermore that we are not obliged at this level to specify the helicity. This expression exhibits the existence of a degeneracy: for each Heisenberg theory characterized by a given value of the self-coupling s there are six distinct class of Dirac spinors, which we could identify to three neutrinos and their corresponding anti-neutrinos. A more careful analysis of the topological sectors allows the identification of three pure states of definite helicity, say $\nu_{WZ} = (\nu_{11}, \nu_{44}, \nu_{41})$ and their corresponding anti-states $(\nu_{22}, \nu_{33}, \nu_{32})$. Note that the interacting neutrinos that appears effectively in real processes are not in general identical to the pure cases. Thus in a very analogous way as is done in case of massive eigenstates of neutrinos, we can write the interacting ones ν_α in terms of the above pure states ν_i by means of a unitary matrix:

$$\nu_\alpha = \mathcal{M}_{\alpha i}\, \nu_i \tag{35}$$

for α and i from 1 to 3; and their corresponding conjugate antistates. In this framework we can understand the change of flavor of massless neutrinos.

A concrete way to improve this statement can be undertaken through the following steps. First, we prove that the Heisenberg spinors satisfying eq. (2) admits a state of free-particle displayed in the plane-wave solution

$$\Psi = e^{ik_\alpha x^\alpha}\, \Psi^o, \tag{36}$$

where Ψ^o is a constant spinor written in terms of two-components spinors:

$$\Psi^o = \begin{pmatrix} \varphi \\ \eta \end{pmatrix},$$

such that

$$\eta = \left(\frac{\sigma_i k^i - 2isB_o}{k_0 - 2sA_o} \right) \varphi. \tag{37}$$

Compatibility requires the "on-mass" condition

$$k_\mu k^\mu = 4s^2\,(A_o^2 - B_o^2).$$

This allows us to write the time development in vacuum of the flavor state $\nu_\alpha = \nu_e, \nu_\mu$ or ν_τ from eq. (35) in the form

$$\nu_\alpha(t) = \mathcal{M}_{\alpha i}\, \nu_i = \mathcal{M}_{\alpha i} \mathcal{M}^*_{\beta i}\, e^{-iE_i t} \nu_\beta, \tag{38}$$

which exhibits the oscillation of flavors

$$i\frac{\mathrm{d}}{\mathrm{d}t}\, \nu_\alpha = \mathcal{S}_\alpha{}^\beta\, \nu_\beta.$$

Thus the above decomposition of Dirac fields in terms of Heisenberg fields allows to understand neutrino oscillation without assuming that such process is a demonstration that neutrinos are massive particles. The present mechanism of description allows to understand oscillations occurring for massless neutrino.

Let us finally state that in addition, changing the value of s allows the decomposition not only of neutrinos but also of other fields in terms of fundamental Heisenberg spinors.

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Appendix. – The basis of the properties needed to analyze non-linear spinors properties are contained in the Pauli-Kofink (PK) relation that establishes a set of tensor relations concerning elements of the four-dimensional

Clifford γ-algebra. For any element Q of this algebra the PK relation states the validity of

$$(\bar{\Psi}Q\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}Q\Psi)\Psi - (\bar{\Psi}Q\gamma_5\Psi)\gamma_5\Psi. \tag{A.1}$$

for Q equal to I, γ^μ, γ_5 and $\gamma^\mu\gamma_5$. As a consequence of this relation we obtain two extremely important consequences:

– The norm of currents J_μ and I_μ has the same strength but opposite signs.

– The vectors J_μ and I_μ are orthogonal.

Indeed, using the PK relation we have

$$(\bar{\Psi}\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}\Psi)\Psi - (\bar{\Psi}\gamma_5\Psi)\gamma_5\Psi.$$

Multiplying by Ψ and using the definitions above it follows

$$J^\mu J_\mu = A^2 + B^2. \tag{A.2}$$

We also have

$$(\bar{\Psi}\gamma_5\gamma_\lambda\Psi)\gamma^\lambda\Psi = (\bar{\Psi}\gamma_5\Psi)\Psi - (\bar{\Psi}\Psi)\gamma_5\Psi.$$

From which it follows that the norm of I_μ is

$$I^\mu I_\mu = -A^2 - B^2 \tag{A.3}$$

and that the four-vector currents are orthogonal

$$I_\mu J^\mu = 0. \tag{A.4}$$

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