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Bouncing cosmologies

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Contents

1.1. 12

1.3.

1.4

2.1.

2.2.

2.3. 2.4. 2.2.1.

1.

2.

3.

ABSTRACT

We review the general features of nonsingular universes (i.e. those that go from an era of accelerated collapse to an expanding era without displaying a singularity) as well as cyclic universes. We discuss the mechanisms behind the bounce, and analyze examples of solutions that implement these mechanisms. Observational consequences of such regular cosmologies are also considered, with emphasis in the behavior of perturbations.

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3	3.1.	. Scalar field in the presence of a potential		142
3	3.2.	Dynamical origin of the geometry		
		3.2.1.	WIST (Weyl Integrable Space Time)	146
		3.2.2.	WIST duality: The Weyl map	147
		3.2.3.	Invariant action in WIST	147
		3.2.4.	A particular case of WIST duality	147
		3.2.5.	A nonsingular cosmological model in WIST	148
		3.2.6.	The WIST function $\varphi(t)$: Structural transitions	149
		3.2.7.	WISTons and anti-WISTons: On the geometrization of instantons	150
		3.2.8.	Weylization	152
		3.2.9.	Solution with matter generation	153
3	3.3.	Scalar-	tensor theories	154
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		3.3.1.	Corrections coming from string theory	156		
		3.3.2.	String pre-big bang	156		
	3.4.	Appen	dix: Conformal transformation	157		
4.	Maxv	vellian ar	nd non-Maxwellian vector fields	159		
	4.1.	Introdu	lction	159		
	4.2.	Einstei	n-Maxwell singular universe	159		
	4.3.	Non-m	inimal interaction	159		
	4.4. An example of a non singular universe			160		
	4.5.	Nonlin	ear electrodynamics			
		4.5.1.	Magnetic universe	162		
		4.5.2.	Born-Infeld electrodynamics			
		4.5.3.	Bouncing in the magnetic universe			
		4.5.4.	Two-fluid description	165		
	4.6.	Appen	dix	166		
		4.6.1.	Repulsive gravity	166		
		4.6.2.	Global dual invariance	167		
5.	Visco	sity				
6.	Boun	ces in the	e braneworld			
7.	Varia	ble cosm	ological constant			
8.	Past-	eternal u	niverses			
	8.1.	Variab	le cosmological constant			
	8.2.	Fundar	nental state for $f(R)$ theories			
~	8.3.	The en	nergent universe			
9.	Quan	tum cosn	nology			
	9.1.	Ine on	tological (Bonm-de Broglie) interpretation	1/b		
	9.2.	Loop q	uantum gravity			
10	9.3.	Stocha	stic approach			
10.	Cyclic universes					
	10.1.	10.1. Thermodynamical arguments				
	10.2.		Chapters in the matter side of FE	102 102		
		10.2.1.	Cualiges in the matter side of EE	102 102		
		10.2.2.	Cyclic universes in loop quantum gravity	103 195		
		10.2.5.	The cyclic universe based on the ekpyretic universe	10J 196		
		10.2.4.	Oscillatory universe from the steady state model	180 187		
		10.2.5.	Other models			
	10.3	Issues	of the cyclic models			
11	Pertu	rhations	in hounging universes	100 188		
11.	11 1	Tetta balons in Dodals				
		11 1 1	Scalar nerturbations	190 191		
		11.1.1.	Vector perturbations in a contracting background			
		11 1 3	Tensor perturbations	192		
	112	Scalar	nerturbations in exact models using the quasi-Maxwellian framework	192		
	11.3. Matching					
	114 Creation of cosmological magnetic fields					
	11.5 Appendix					
	11101	11.5.1	Perturbations using Bardeen variables	200		
		11.5.2.	The quasi-Maxwellian method			
	11.6.	Relatio	n between the two methods			
12.	Concl	usion				
•	Ackno	Acknowledgements				
	References					

The world, an entity out of everything, was created by neither gods nor men, but was, is and will be eternally living fire, regularly becoming ignited and regularly becoming extinguished. Heraclitus.¹

¹ The proof of this assertion – which is still missing – was left by Heraclitus to future generations.

1. Introduction

The standard cosmological model (SCM) (see for instance [420] for an updated review) furnishes an accurate description of the evolution of the universe, which spans approximately 13.7 billion years. The main hypothesis on which the model is based are the following:

- 1. Gravity is described by General Relativity.
- 2. The universe obeys the Cosmological Principle [109]. As a consequence, all the relevant quantities depend only on global Gaussian time.
- 3. Above a certain scale, the matter content of the model is described by a continuous distribution of matter/energy, which is described by a perfect fluid.

In spite of its success, the SCM suffers from a series of problems such as the initial singularity, the cosmological horizon, the flatness problem, baryon asymmetry, and the nature of dark matter and dark energy.^{2,3} Although inflation (which for many is currently a part of the SCM) partially or totally answers some of these, it does not solve the crucial problem of the initial singularity [69,70].⁴ The existence of an initial singularity is disturbing: a singularity can be naturally considered as a source of lawlessness [142], because the spacetime description breaks down "there", and physical laws presuppose spacetime. Regardless of the fact that several scenarios have been developed to deal with the singularity issue, the breakdown of physical laws continues to be a conundrum after almost a hundred years since the discovery of the FLRW solution⁵ (which inevitably displays a past singularity, or in the words of Friedmann [166], a *beginning of the world*).

In this review, we shall concentrate precisely on the issue of the initial singularity.⁶ We will see that non-singular universes have been recurrently present in the scientific literature. In spite of the fact that the idea of a cosmological bounce is rather old, the first exact solutions for a bouncing geometry were obtained by Novello and Salim [310], and Melnikov and Orlov [286] in the late 70's.⁷ It is legitimate to ask why these solutions did not attract the attention of the community then. In the beginning of the 80's, it was clear that the SCM was in crisis (due to the problems mentioned above, to which we may add the creation of topological defects, and the lack of a process capable of producing the initial spectrum of perturbations, necessary for structure formation). On the other hand, at that time the singularity theorems were taken as the last word about the existence of a singularity in "reasonable" cosmological models. The appearance of inflationary theory gave an answer to some of the issues in a relatively economical way, and opened the door for an explanation of the origin of the spectrum of primordial fluctuations. Faced with these developments, and taking into account the status of singularity theorems at that time, the issue of the initial singularity was not pressing anymore, and was temporarily abandoned in the hope that quantum gravity would properly address it. At the end of the 90's, the discovery of the acceleration of the universe brought back to the front the idea that $\rho + 3p$ could be negative, which is precisely one of the conditions needed for a cosmological bounce in GR, and contributed to the revival of nonsingular universes. Bouncing models even made it to the headlines in the late 90's and early XXI century, since some models in principle embedded in string theory seemed to suggest that a bouncing geometry could also take care of the problems solved by inflation.

Perhaps the main motivation for nonsingular universes is the avoidance of lawlessness, as mentioned above.⁸ Also, since we do not know how to handle infinite quantities, we would like to have at our disposal solutions that do not entail divergencies. As will be seen in this review, this can be achieved at a classical level, and also by quantum modifications. On a historical vein, this situation calls for a parallel with the status of the classical theory of the electron by the end of the 19th century. The divergence of the field on the world line of the electron led to a deep analysis of Maxwell's theory, including the acceptance of a cooperative influence of retarded and advanced fields [355],⁹ and the related causality issues. However, this divergence is milder than that of some solutions of General Relativity, since it can be removed by the interaction of the electron with the environment. Clearly, this is not an option when the singularity is that of a cosmological model.

Another motivation for the elimination of the initial singularity is related to the Cauchy problem. In the SCM, the structure of spacetime has a natural foliation (if no closed timelike curves are present), from which a global Gaussian coordinate system

² There are even claims that standard cosmology does not predict the value of the present CMBR temperature [215].

³ Some "open questions" may be added to this list, such as why the Weyl tensor is nearly null, and what the future of the universe is.

⁴ Inflation also presents some problems of its own, such as the identification of the inflaton with a definite field of some high-energy theory, the functional form of the potential *V* in terms of the inflaton [52], and the need of particular initial conditions [181]. See also [302].

⁵ This acronym refers to the authors that presented for the first time the solution of EE that describes a universe with zero pressure (Friedmann [166]) and nonzero pressure (Lemâitre [253]), and to those who studied its general mathematical properties and took it to its current form (Robertson [353] and Walker [412]). For historical details, see [288].

 $^{^6}$ We shall not analyze the existence of future singularities, such as the so-called sudden future singularities [36] or the "Big Rip" [92].

⁷ An approximate bouncing solution for a massive minimally coupled scalar field in General Relativity was presented in [383].

⁸ It is worth noting that Einstein was well aware of the problem of singularities in GR [336], and he made several attempts to regularize some solutions of his theory, such as the so-called Einstein-Rosen bridge, in the early 30s. Indeed, he wrote "The theory (GR) is based on a separation of the concepts of the gravitational field and matter. While this may be a valid approximation for weak fields, it may presumably be quite inadequate for very high densities of matter. One may not therefore assume the validity of the equations for very high densities and it is just possible that in a unified theory there would be no such singularity" [146].

⁹ In fact, it can be said that the problem of the singularity of the classical theory of the electron was transcended, if not resolved, by the quantization of the EM field.

can be constructed, with $g_{00} = 1$, $g_{0i} = 0$, in such a way that

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{g}_{ii}\,\mathrm{d}x^i\mathrm{d}x^j.$$

The existence of a global coordinate system allows a rigorous setting for the Cauchy problem of initial data. However, it is the gravitational field that diverges on a given spatial hypersurface t = const. (denoted by Σ) at the singularity in the SCM. Hence, the Cauchy problem cannot be well formulated on such a surface: we cannot pose on Σ the initial values for the field to evolve.

There are more arguments that suggest that the singularity should be absent in an appropriate cosmological model. According to [48], the second law of thermodynamics is to be supplemented with a limit on the entropy of a system of largest linear dimension *R* and proper energy *E*, given by

$$\frac{S}{E} \leq \frac{2\pi R}{\hbar c}.$$

Currently this bound is known to be satisfied in several physical systems [369]. It was shown in [49] that the bound is violated as the putative singularity is approached in the radiation-dominated FLRW model (taking as *R* the particle horizon size). The restriction to FLRW models was lifted in [369], where it was shown, independently of the spacetime model, and under the assumptions that (1) causality and the strong energy condition (SEC, see Appendix) hold, (2) for a given energy density, the matter entropy is always bounded from above by the radiation entropy, that the existence of a singularity is inconsistent with the entropy bound: a violation occurs at time scales of the order of Planck's time.¹⁰

From the point of view of quantum mechanics, we could ask if it is possible to repeat in gravitation what was done to eliminate the singularity in the classical theory of the electron. Namely, can the initial singularity be smoothed via quantum theory of gravity? The absence of the initial singularity in a quantum setting is to be expected on qualitative grounds. There exists only one quantity with dimensions of length that can be constructed from Newton's constant *G*, the velocity of light *c*, and Planck's constant \hbar (namely Planck's length $\ell_{Pl} = \sqrt{G\hbar/c^3}$). This quantity would play in quantum gravity a role analogous to that of the energy of the ground state of the hydrogen atom (which is the only quantity with dimensions of energy that can be built with fundamental constants) [62]. As in the hydrogen atom, ℓ_{Pl} would imply some kind of discreteness, and a spectrum bounded from below, hence avoiding the singularity.¹¹ Also, since it is generally assumed that ℓ_{Pl} sets the scale for quantum gravity effects, geometries in which curvature can become larger than ℓ_{Pl}^{-2} or can vary very rapidly on this scale would be highly improbable.

Yet another argument that suggests that quantum effects may tame a singularity is given by the Rayleigh–Jeans spectrum. According to classical physics, the spectral energy distribution of radiation in thermal equilibrium diverges like ω^3 at high frequencies, but when quantum corrections are taken into account, this classical singularity is regularized and the Planck distribution applies [177]. We may expect that QG effects would regularize the initial singularity.

As a consequence of all these arguments indicating that the initial singularity may be absent in realistic descriptions of the universe, many cosmological solutions displaying a bounce were examined in the last decades. In fact, the pattern in scientific cosmologies somehow parallels that of the cosmogonic myths in diverse civilizations, which can be classified in two broad classes. In one of them, the universe emerges in a single instant of creation (as in the Jewish–Christian and the Brazilian Carajás cosmogonies [116]). In the second class, the universe is eternal, consisting of an infinite series of cycles (as in the cosmogonies of the Babylonians and Egyptians) [381].

We have seen that there are reasons to assume that the initial singularity is not a feature of our universe. Quite naturally, the idea of a non-singular universe has been extended to encompass cyclic cosmologies, which display phases of expansion and contraction. The first scientific account of cyclic universes is in the papers of Friedmann [268], Einstein [147], Tolman [395], and Lemaĭtre [254] and his Phoenix universe, all published in the 1930's. A long path has been trodden since those days up to recent realizations of these ideas (as for instance [179], see Section 10.2.4). We shall see in Section 10 that some cyclic models could potentially solve the problems of the standard cosmological model, with the interesting addition that they do not need to address the issue of the initial conditions.

Another motivation to consider bouncing universes comes from the recognition that a phase of accelerated contraction can solve some of the problems of the SCM in a manner similar to inflation. Let us take for instance the flatness problem (see also Section 10). Present observations imply that the spatial curvature term, if not negligible, is at least non-dominant w.r.t. the curvature term:

$$r^2=rac{|\epsilon|}{a^2H^2}\lesssim 1,$$

but during a phase of standard, decelerated expansion, r grows with time. Indeed, if $a \sim t^{\beta}$, then $r \sim t^{1-\beta}$. So we need an impressive fine-tuning at, say, the GUT scale, to get the observed value of r.¹² This problem can be solved by introducing an early phase during which the value of r, initially of order 1, decreases so much in time that its subsequent growth during

¹⁰ For an updated discussion of the several types of entropy bounds in the literature, see [86].

¹¹ This expectation has received support from the proof that the spectrum of the volume operator in LQG is discrete, see for instance [267].

¹² But notice that the flatness problem may actually not be a problem at all if gravity is not described by GR, see Section 2.2.

FLRW evolution keeps it still below 1 today. This can be achieved by [179] power-law inflation ($a \sim t^{\beta}, \beta > 1$), pole inflation ($a \sim (-t)^{\beta}, \beta < 0, t \rightarrow 0_{-}$), and accelerated contraction ($0 < \beta < 1, t \rightarrow 0_{-}$) [172]. Thus, an era of accelerated contraction may solve the flatness problem (and the other kinematical issues of the SCM [179]). This property helps in the construction of a scenario for the creation of the initial spectrum of cosmological perturbations in non-singular models (see Section 11).

The main goal of this review is to present some of the many non-singular solutions available in the literature, exhibit the mechanism by which they avoid the singularity, and discuss what observational consequences follow from these solutions and may be taken (hopefully) as an unmistakable evidence of a bounce. We shall not pretend to produce an exhaustive list, but we intend to include at least an explicit form for the time evolution of a representative member of each type of solution.¹³ The models examined here will be restricted to those close or identical to the FLRW geometry.¹⁴ Although theories other than GR will be examined, we shall not consider multidimensional theories (exception made for models derived from string theory, see Section 3.3.2) or theories with torsion.

We shall start in Section 1.2 by stating a working definition of a nonsingular universe, and giving a brief account of the criteria that can be used to determine whether a certain model is singular or not. It will suffice for our purposes in this review to define a singularity as the region where a physical property of the matter source or the curvature "blows up" [411]. In fact, since we shall be dealing almost exclusively with geometries of the Friedmann type, the singularity is always associated with the divergence of some functional of the curvature.¹⁵

Let us remark at this point that there are at least two different types of nonsingular universes: (a) bouncing universes (in which the scale factor attains a minimum), and (b) "eternal universes", which are past infinity and ever expanding, and exist forever. Class (a) includes cyclic universes. The focus of this review are those models in class (a), although we shall review a few examples of models in class (b) in Section 8.

1.1. Notation, conventions, etc

Throughout this report, the Einstein's equations (EE) are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu},$$

where Λ is the cosmological constant, and $\kappa = 8\pi G/c^4$, which we shall set equal to 1, unless stated otherwise, while the metric of the FLRW model is

$$ds^{2} = dt^{2} - a^{2}(t) \left[\frac{dr^{2}}{1 - \epsilon r^{2}} + r^{2}(d\theta^{2} + \sin^{2}\theta d\varphi^{2}) \right],$$
(1)

where $\epsilon = -1, 0, +1$. The 3-dimensional surface of homogeneity t = constant is orthogonal to a fundamental class of observers endowed with a four-velocity vector field $v^{\mu} = \delta^{\mu}_{0}$. In the case of a perfect fluid with energy density ρ and pressure p, EE take the form

$$\dot{\rho} + 3(\rho + p)\frac{a}{a} = 0, \tag{2}$$

$$\frac{\dot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3},$$
(3)

in which Λ is the cosmological constant, and the dot denotes the derivative w.r.t. cosmological time. These equations admit a first integral given by the so-called Friedmann equation:

$$\frac{1}{3}\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{\epsilon}{a^2} - \frac{\Lambda}{3}.$$
(4)

The energy–momentum tensor of a theory specified by Lagrangian \mathcal{L} is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}},\tag{5}$$

where $g = det(g_{\mu\nu})$. In the case of a perfect fluid, $T_{\mu\nu}$ takes the form

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu},$$

where u_{μ} is the velocity of the fluid.

¹³ The issue of singularities in cosmology has been previously dealt with in [168].

¹⁴ Notice however the solutions given in [120,370]. These are non-singular but do not display the symmetries of the observed universe, although they are very useful as checks of general theorems.

¹⁵ But notice that not all types of singularities have large curvature, and diverging curvature is not the basic mechanism behind singularity theorems. If we consider the problem of singularities in a broad sense, we seem to be "treating a symptom rather than the cause" when addressing exclusively unbounded curvature [66].

1.2. Singularities, bounces, and energy conditions

The issue of the initial singularity of the FRLW solution was debated for a long time, since it was not clear if this singular state was an inherent trace of the universe or just a consequence of the high degree of symmetry of the model. This question was discussed firstly in an analytical manner by Lifshitz and collaborators in [50], where geometries that are solutions of EE with a maximum number of allowed functions were analyzed. The results wrongly suggested that the singularity was not unavoidable, but a consequence of the special symmetries of the FLRW solution.¹⁶

From a completely different point of view, Hawking, Penrose, Geroch and others developed theorems that give global conditions under which timelike and null geodesics cannot be extended beyond a certain (singular) point [142,206]. The goal in this case was not about proving the existence of a region of spacetime in which some functional of the metric is divergent. Instead, the issue of the singularity was considered from a wider perspective, characterizing a spacetime as a whole, by way of its global properties, such as the abrupt termination of some geodesics in the manifold. Let us present a typical example of these theorems [205]:

Theorem. The following requirements cannot all be true for a given space-time \mathfrak{M}

- 1. There exists a compact spacelike hypersurface (without boundary) 55;
- 2. The divergence θ of the unit normals to \mathfrak{H} is positive at every point of \mathfrak{H} ;
- 3. $R_{\mu\nu} v^{\mu} v^{\nu} \leq 0$ for every non-spacelike vector v^{μ} ;
- 4. mis geodesically complete in past timelike directions.

Notice that the link of this theorem with physics comes through condition (3) via EE, yielding a statement about the energy-momentum tensor:

$$T_{\mu\nu}\nu^{\mu}\nu^{\nu} - \frac{T}{2} \ge 0, \tag{6}$$

called the strong energy condition (SEC), see the Appendix. Notice also that, although not explicitly mentioned, this theorem assumes the absence of closed timelike curves [142]. With hindsight,¹⁷ it can be said that the strength of these theorems is the generality of their assumptions (at the time they were conceived), while their weakness is that they give little information about how the approach to the singularity is described in terms of the dynamics of the theory or about the nature of the singularity. In any case, if we assume that the universe is nonsingular, a positive attitude regarding the singularity theorems is to consider that they show the limits of applicability of "reasonable" hypothesis (such as GR or the energy conditions, see the Appendix) [66].

A local definition of a bounce can also be given, in the GR framework, in terms of the so-called Tolman wormhole [291,210] (see below). Both in this case and in that of the above mentioned theorems, the non-singular behavior in GR is only possible when the SEC is violated. The assumption of such a condition seemed reasonable in the early seventies, but several situations have been examined in the literature that may be relevant in some epoch of the evolution of the universe, for which SEC is not fulfilled, such as curvature-coupled scalar fields and cosmological inflation [26,27,291,356].

Next we shall examine in some detail how the singularity can be avoided. In the following, we shall use a simple form of the singularity theorems.¹⁸ Let us first introduce some definitions (following [150]). The covariant derivative of the 4-velocity v_{μ} of the fluid that generates the geometry can be decomposed as follows

$$\mathbf{v}_{\nu;\mu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\nu\mu} + \mathbf{v}_{\mu}\dot{\mathbf{v}}_{\nu},\tag{7}$$

where $\theta = v^{\mu}_{;\mu}$ is the expansion, $h_{\mu\nu} = g_{\mu\nu} - v_{\mu}v_{\nu}$, the trace-free symmetric shear tensor is denoted by $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ is the vorticity tensor (see Eqs. (363) and (364)). Defining S by¹⁹

$$\frac{\dot{S}}{S} = \frac{\theta}{3},\tag{8}$$

the Raychaudhuri equation [349], which follows from Eq. (7) can be written as²⁰

$$3\frac{\ddot{S}}{S} + 2(\sigma^2 - \omega^2) - \dot{v}^{\mu}_{;\mu} = -\frac{1}{2}(\rho + 3p) + \Lambda,$$
(9)

where $A_{\mu} = v^{\nu}v_{\mu;\nu} \equiv \dot{v}^{\mu}$ is the acceleration.

¹⁶ For a reappraisal of the work in [50], see for instance [351] and references therein.

¹⁷ From a mathematical point of view, a negative energy could also allow for a bounce. We will not examine this possibility in the present paper.

¹⁸ This will suffice for our goals, more refined formulations can be found in [370].

¹⁹ *s* corresponds to the scale factor *a* in the case of the FLRW universe.

²⁰ This equation was independently obtained by Komar [248].

Theorem ([148]). In a universe where $\rho + 3p \ge 0$ is valid, $\Lambda \le 0$, and $\dot{v}^{\mu} = \omega^{\mu\nu} = 0$ at all times, at any instant when $H = \frac{1}{3}\theta > 0$, there must have been a time $t_0 < 1/H$ such that $S \to 0$ as $t \to t_0$. A space–time singularity occurs at $t = t_0$, in such a way that ρ and the temperature T diverge.

Several remarks are in order. First, EE were used to obtain Eq. (9). Hence, the consequences of the theorem are only valid in the realm of GR. Second, the singularity implied in the theorem is universal: any past-directed causal curve ends at it with a finite proper length, in line with a coherent definition of a cosmological singularity (if null curves are allowed for causal curves, then affine length has to be used for them instead of proper length which would vanish).²¹ Third, since there is no restriction on the symmetries of the geometry, θ is in principle a function of all the coordinates, so that the theorem applies not only to Friedmann–Lemaitre–Robertson–Walker (FLRW) models, but also to most of the spatially homogeneous, and to some inhomogeneous models (see examples in [370]). Fourth, as we mentioned before, the condition $\rho + 3p \ge 0$, or more generally, SEC, is violated even at the classical level, for instance by the massive scalar field, and also at the quantum level (as in the Casimir effect²²). So it would be desirable to have singularity theorems founded on more general energy conditions, but this goal has not been achieved yet (see [370]).

Notice that in the general case, acceleration and/or rotation could in principle avoid the singularity [370], but high pressure cannot prevent the initial singularity in the FLRW model. Rather, it accelerates the collapse. This can be seen as follows. The conservation equations $T^{\mu}_{\ \nu\mu} = 0$ give

$$v^{\mu}\rho_{,\mu} + (\rho + p)\theta = 0,$$

 $(\rho + p)A^{\mu} = -h^{\mu\nu}p_{,\nu},$

Since $p_{,i} = 0$ in the FLRW, there is no acceleration. Furthermore, pressure contributes to the the active gravitational mass $\rho + 3p$. Finally, not even a large and positive Λ can prevent the singularity in the context of the theorem [148].

As mentioned before, a bounce can also be defined locally. The minimal conditions from a local point of view for a bounce to happen in the case of a FLRW universe were analyzed in [291], where a Tolman wormhole was defined as a universe that undergoes a collapse, attains a minimum radius, and subsequently expands. Adopting in what follows the metric equation (1), to have a bounce it is necessary that $\dot{a}_b = 0$, and $\ddot{a}_b \ge 0$. For this to be a true minimum of the scale factor (conventionally located at t = 0) there must exists a time \tilde{t} such that $\ddot{a} > 0$ for all $t \in (-\tilde{t}, 0) \cup (0, \tilde{t})$. From EE in the FLRW universe (neglecting the cosmological constant term) we get

$$\begin{split} \rho &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2}\right),\\ p &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2}\right). \end{split}$$

From these, the combinations relevant for the energy conditions (see Section 1.4) are:

$$\rho + p = 2\left(-\frac{d^2\ln a}{dt^2} + \frac{\epsilon}{a^2}\right),$$

$$\rho - p = 2\left(\frac{1}{3a^3}\frac{d^2(a^3)}{dt^2} + 2\frac{\epsilon}{a^2}\right),$$

$$\rho + 3p = -6\frac{\ddot{a}}{a}.$$

From these conditions and $\dot{a}_b = 0$, and $\ddot{a}_b \ge 0$ it follows that [291]

 \exists bounce and $\epsilon = -1 \Rightarrow$ NEC violated, ²³

 \exists bounce and ($\epsilon = 0$; $\ddot{a}_b > 0$) \Rightarrow NEC violated,

 \exists bounce and ($\epsilon = 1$; $\ddot{a}_b > a_b^{-1}$) \Rightarrow NEC violated.

The definition of ρ and p and $\ddot{a} > 0$ imply that:

$$\begin{split} \rho + p &< 2\frac{\epsilon}{a^2},\\ \rho - p &> 2\frac{\epsilon}{a^2},\\ \rho - 3p &< 0. \end{split}$$

²¹ See [370,102] for a classification of singularities.

²² In fact, it has been shown in [208] that the Casimir effect associated to a massive scalar field coupled to the Ricci scalar in a closed universe can lead to a bounce.

²³ For the energy conditions, see Section 1.4.

It follows that

 \exists bounce and $\epsilon \neq 1 \Rightarrow$ NEC violated,

 \exists bounce \Rightarrow SEC violated.

The case that minimizes the violations of the energy conditions can be stated as

 \exists bounce and $(\epsilon = +1; \ddot{a}_b \le a_b^{-1}) \Rightarrow$ NEC, WEC, DEC satisfied; SEC violated.

This result may be expected since the curvature term with $\epsilon = +1$ acts like a negative energy density in Friedmann's equation. Notice that in this analysis, only Einstein's equations and the point-wise energy conditions were used, without assuming any particular equation of state. In a certain sense, this is the inverse of the theorem stated earlier, which assumed the validity of the SEC.²⁴

The restriction to a FLRW model was lifted in a subsequent paper [210], and the analysis in a general case was done following standard techniques taken from the ordinary wormhole case [209]. It was found that even in the case of a geometry with no particular symmetries, the SEC must be violated if there is to be a bounce in GR. Consequently, one can conclude that the singularity theorems that assume that SEC is valid cannot be improved. A highlight in these analysis is that only the local geometrical structure of the bounce was needed; no assumptions about asymptotic or topology were required, in contrast with the Hawking–Penrose singularity theorems [371]. Equally important is the fact that, as mentioned above, SEC may not be such a fundamental physical restriction.

To summarize what was discussed up to now, we can say that there is a "window of opportunity" to avoid the initial singularity in FLRW models at a classical level by one or a combination of the following assumptions²⁵:

- 1. Violating SEC in the realm of GR²⁶;
- 2. Working with a new gravitational theory, as for instance those that add scalar degrees of freedom to gravity (Brans–Dicke theory being the paradigmatic example of this type, see Section 3), or by adopting an action built with higher-order invariants (see Section 2).

As will be seen below, other ways to avoid the singularity are:

- 1. Changing the way gravity couples to matter (from minimal to non-minimal coupling, see for instance the case of the scalar field in Section 3) (see for instance [187]);
- 2. Using a non-perfect fluid as a source, see Section 5.

Finally, quantum gravitational effects also give the chance of a bounce (see Section 9.2).²⁷

1.3. Extrema of a(t) and $\rho(t)$

Let us study the relations imposed by EE between extrema of the scale factor, the energy density, and the energy conditions, in the case of one fluid. Let us recall that the sufficient conditions to have a bounce $\operatorname{are}^{28} \theta_b = 0$ and $\dot{\theta}_b > 0$, where $\theta = 3\dot{a}/a$, and the subindex *b* denotes that the quantities are evaluated at the bounce. It follows from Raychaudhuri's equation for the FLRW model (Eq. (9)) with $\Lambda = 0$,

$$\dot{\theta} + \frac{\theta^2}{3} = -\frac{1}{2}(\rho + 3p),\tag{10}$$

that at the bounce we must have $(\rho + 3p)|_b < 0$, independently of the value of ϵ (as was also shown in the previous section). From the conservation equation,

 $\dot{\rho} = -(\rho + p)\theta,$

we see that there may be extrema of ρ when $\theta_e = 0$ (as in the case of a putative bounce) and/or when $\rho_e = -p_e$. The second derivative of the energy density is given by

$$\ddot{\rho} = -(\dot{\rho} + \dot{p})\theta - (\rho + p)\theta. \tag{11}$$

Let us assume first that $\theta_e = 0$ with $\rho_e + p_e \neq 0$, which implies that $\dot{\rho}_e = 0$ and

$$\ddot{
ho}_e=-(
ho_e+p_e)\dot{ heta}_e,\qquad \dot{ heta}_e=-rac{1}{2}(
ho_e+3p_e).$$

The different possibilities, according to the sign of $\dot{\theta}_e$, $\rho_e + p_e$, and $\rho + 3p$ are displayed in the Table 1.1:

²⁸ We are assuming that $\ddot{a} \neq 0$.

²⁴ An analysis along the same lines but with a more general parametrization for the scale factor was carried out in [339].

²⁵ We shall not consider here the existence of closed timelike curves as a possible cause of a nonsingular universe.

 $^{^{26}}$ A complete analysis of the behavior of the energy conditions for different types of singularities has been presented in [102].

²⁷ A definition of a nonsingular space using the so-called principle of quantum hyperbolicity has been given in [66].

Table	1.1
-------	-----

$ \rho_e + 3p_e $	$\dot{ heta_e}$	$ ho_e + p_e$	$\ddot{ ho}_e$	$ ho_e$	a _e
<0	>0	<0	>0	min.	min.
	20	>0	<0	max.	
>0	<0	<0	<0	max.	max.
		>0	>0	mín.	

We see that there are two cases that agree with what may be termed "normal matter" (rows 2 and 4), in the sense that maximum (minimum) compression leads to maximum (minimum) energy density. Notice however that the case in row 2 violates the strong energy condition (see Appendix). The other cases are clearly unusual: minimum density with minimum scale factor (row 1), and the opposite (that is, maximum density with maximum scale factor, row 3).²⁹ Notice that it is the null energy condition $\rho + p > 0$ (see Appendix) and not the SEC that is violated at these unusual cases. In fact, if the requirement $\rho + p \ge 0$ is not satisfied, then the equation of energy conservation for a perfect fluid,

$$\dot{\rho} = -\theta(\rho + p),\tag{12}$$

says that compression would entail a decreasing energy density, which is a rather unexpected behavior for a fluid.³⁰ Examples of the four behaviors will be found along this review.

When an EOS $p = \lambda \rho$ plus the condition $\rho > 0$ are imposed,³¹ we see that the case in row 1 is permitted for $\lambda < -1$, and that in row 2, for $\lambda \in (-1, -1/3)$. The case in row 3 is not allowed for any λ , while that in row for is permitted for $\lambda > -1/3$.

Notice that all the extrema in ρ in Table 1.1 are global, since the other possibility (given by $\rho_e + p_e = 0$) leads to an inflection point in ρ , assuming that $p = \lambda \rho$.

1.4. Appendix: Energy conditions

We shall give next the general expression of the energy conditions, and also their form for the particular case of the energy-momentum tensor given by

$$T^{\mu}_{\nu} = \text{diag}(\rho, -p, -p, -p).$$
 (13)

• The null energy condition (NEC) states that for any null vector,

$$NEC \Leftrightarrow T_{\mu\nu}k^{\mu}k^{\nu} \ge 0.$$
⁽¹⁴⁾

In terms of Eq. (13),

$$NEC \Leftrightarrow \rho + p \ge 0. \tag{15}$$

• The weak energy condition (WEC) asserts that

$$WEC \Leftrightarrow T_{\mu\nu} v^{\mu} v^{\nu} \ge 0 \tag{16}$$

for any timelike vector. In terms of Eq. (13),

$$\rho \ge 0, \quad \text{and} \quad \rho + p \ge 0.$$
 (17)

• The strong energy condition (SEC) is the assertion that, for any timelike vector,

$$SEC \Leftrightarrow \left(T_{\mu\nu} - \frac{l}{2}g_{\mu\nu}\right)v^{\mu}v^{\nu} \ge 0.$$
(18)

In terms of Eq. (13),

$$\rho + p \ge 0, \quad \text{and} \quad \rho + 3p \ge 0. \tag{19}$$

Each of these three conditions has an averaged counterpart [410]. There is yet another condition:

• The dominant energy condition (DEC) says that for any timelike vector

$$DEC \Leftrightarrow T_{\mu\nu}v^{\mu}v^{\nu} \ge 0$$
 and $T_{\mu\nu}v^{\nu}$ is not spacelike. (20)

²⁹ The former is precisely the behavior that allows for a bounce in loop quantum gravity [60] (see Section 9.2), while the latter is what is found in the so-called big-rip [284].

³⁰ Fluids that violate the NEC are called *phantom* or *ghost* fluids, and have been studied in [91].

³¹ Notice that some models do not satisfy this conditions, see for instance Eq. (184).

The different energy conditions are not independent. The following relations are valid:

$WEC \Rightarrow NEC$,	(21)
$SEC \Rightarrow NEC$,	(22)
$DEC \Rightarrow WEC.$	(23)

Notice that if NEC is violated then all the other pointwise energy conditions would be violated [410].

2. Higher-order gravitational theories

Higher-order terms in the action for gravity (such as R^2 , $R_{\mu\nu}R^{\mu\nu}$, etc.) typically appear due to quantum effects, either in the case of quantized matter in a fixed gravitational background [38,123], or in the gravitational effective action as corrections from quantum gravity [132] or string theory³² [121]. These terms are expected to be important in situations of high curvature, when the scale factor is small.³³ The models that are engineered to work in the intermediate regime, where quantized matter fields evolve on a given classical geometry (the so-called semiclassical approximation) mirror the path taken in the early days of quantum field theory, in which quantum matter was in interaction with a classical electromagnetic background field. In the case of gravity, it is generally agreed that this approach may be valid for distances above ℓ_{Pl} , although this statement can only be verified by a complete quantum theory of gravitation, not yet available. As we shall see in Section 9, some models go below ℓ_{Pl} , incorporating effects expected to be present in the complete theory, but for the time being the quest of the "correct theory" at this energy level seems far from being settled.

2.1. Quantized matter on a fixed background

Let us start by considering the corrections coming from quantum matter in a given background. As shown for instance in [402], in the models based on the semiclassical approximation the mean value of the stress-energy tensor $T_{\mu\nu}$ of a set of quantized fields interacting with a classical geometry is plagued with infinities. These divergencies can be removed by a suitable modification of EE that follows from a renormalization procedure. In order to render the mean value of $T_{\mu\nu}$ finite, the cosmological constant Λ and Einstein's constant κ are renormalized, and a counterterm of the form

$$\Delta L = \sqrt{-g}(\alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}) \tag{24}$$

must be introduced in the Lagrangian.³⁴ The corrections arise from the ultraviolet behavior of the field modes, which only probe the local geometry, hence the appearance of geometric quantities. After the elimination of the divergences and with a convenient choice of α and β , EE with $\langle T_{\mu\nu} \rangle$ as a source preserve their form [402]:

$$G_{\mu\nu} + \Lambda^{(\text{ren})} g_{\mu\nu} = -\kappa^{(\text{ren})} \langle T^{(\text{ren})}_{\mu\nu} \rangle.$$
⁽²⁵⁾

Note that such renormalization does not affect the conservation of the energy-momentum tensor, that is

$$\langle T_{(\text{ren})}^{\mu\nu} \rangle_{;\nu} = 0.$$
 (26)

Notice that there is a residual freedom in the constants introduced by the counterterm [300], so they can be chosen in such a way that they cancel the divergencies without eliminating the quadratic contribution to EE (contrary to what was done in [402]). This more general choice amounts to shifting the constants as $\alpha \rightarrow \alpha + \eta$ and $\beta \rightarrow \beta + \gamma \eta$ [300]. The new equations are

$$G_{\mu\nu} + \eta(\chi_{\mu\nu} + \gamma Z_{\mu\nu}) + \Lambda^{(\text{ren})}g_{\mu\nu} = -\kappa^{(\text{ren})}\langle T^{(\text{ren})}_{\mu\nu}\rangle,$$
(27)

where

$$\frac{1}{2}\chi_{\mu\nu} \equiv R\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right) + R_{;\mu;\nu} - g_{\mu\nu}\Box R,$$
(28)

and

$$Z_{\mu\nu} \equiv R_{;\mu;\nu} - \Box R_{\mu\nu} - \frac{1}{2} (\Box R + R_{\alpha\beta} R^{\alpha\beta}) g_{\mu\nu} + 2R^{\alpha\beta} R_{\alpha\mu\beta\nu}.$$
⁽²⁹⁾

³² Since in this case the non-linear terms are always coupled to one or more scalar fields we shall consider it in Section 3.3.1.

 $^{^{33}}$ As opposed to Lagrangians that are negative powers of *R*, which are currently being considered as candidates to explain the acceleration of the universe [307].

³⁴ The relevance of this type of series development was discussed also by Sakharov [363].

Cosmological solutions of Eq. (27) in the case of the FLRW metric were studied in [301] (see also [396]). For a flat universe, the equations take the form

$$3\left(\frac{\dot{a}}{a}\right)^{2} + 3t_{c}^{2}\left\{\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^{2}\right)\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^{2}\right) - 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^{2}\right)^{2}\right\} = \rho,$$
(30)

$$\dot{\rho} + 3\left(\frac{a}{a}\right)(\rho + p) = 0,\tag{31}$$

where the characteristic time $t_c \equiv 1/\sqrt{c|\mu^2|}$ signals the moment in which the corrections play an important role, and $\mu^{-2} \equiv -2\eta(\gamma + 3)$ (it has dimensions of L^2). For the case of radiation ($\rho = \rho_c a_c^4/a^4$), we get

$$H^{2} + t_{c}^{2} \left\{ \left(\frac{\ddot{a}}{a} - H^{2} \right)^{2} - 2H \left(\frac{\ddot{a}}{a} - H^{3} \right) \right\} = \frac{\rho_{c}}{3} \left(\frac{a_{c}}{a} \right)^{4}, \tag{32}$$

where $H = \dot{a}/a$, and $\rho_c = \rho(t_c)$. If we impose the existence of a bounce by the conditions $a_b > 0$, $\dot{a}_b = 0$, and $\ddot{a}_b > 0$, it follows from this equation that $\mu^{-2} > 0$. It as also shown in [301] $t_c \le 3.33 \times 10^{-4}$ s in order that the theory does not conflict with the three classical tests of GR.

Vacuum solutions of Eq. (27) in the FLRW geometry were studied in [296]. Notice that taking the trace of Eq. (27) in the absence of matter we obtain

$$\ddot{R} + h\dot{R} + \sigma R = 0,$$

where $\sigma = 1/(2\eta(1 + \gamma))$, $h = d[\ln(-g)^{1/2}]/dt$. This equation is analogous to that of a damped harmonic oscillator. Depending on the sign of the parameter σ , and considering h > 0, there may be damped oscillations for R around R = 0, or exponentially decaying or growing solutions [296].

Corrections coming from one-loop contributions of conformally-invariant matter fields on a FLRW background were studied in [384] (see also [164]). They allow for nonsingular solutions that are not of the bouncing type since they describe a universe starting from a deSitter state. A thorough analysis of this setting was given in [15], where the back-reaction problem for conformally invariant free quantum fields in FLRW spacetimes with radiation was studied, for both zero [15] and non-zero [16] curvature and/or Λ . It was found that depending on the values of the regularization parameters, there are some bouncing solutions that approach FLRW at late times.

2.2. Lagrangians depending on the Ricci scalar

On approaching the singularity, powers of the curvature may be expected to play an important dynamical role, hence other possible nonlinear Lagrangians are those belonging to the class defined by

$$S = \int \sqrt{-g} f(R) d^4x, \tag{33}$$

where f(R) is an arbitrary function of the curvature scalar, encompassing polynomials as a particular case.³⁵ The problem of the singularity using this type of Lagrangians has been repeatedly discussed in the literature (see for instance [88,31]). The EOM that follows from this action is

$$f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \Box fg_{\mu\nu} + f'_{,\mu;\nu} = 0,$$
(34)

where $f' \equiv df/dR$. This equation can be expressed in *f* and its derivatives as

$$f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + f''(R_{,\mu;\nu} - \Box Rag_{\mu\nu}) + f'''(R_{,\mu}R_{,\nu} - R_{,\lambda}R^{,\lambda}g_{\mu\nu}) = 0,$$
(35)

or, using the trace,

$$f'\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right) + f''\left(R_{,\mu;\nu} - \frac{1}{4}g_{\mu\nu}\Box R\right) + f'''\left(R_{,\mu}R_{,\nu} - \frac{1}{4}R_{,\lambda}R^{,\lambda}g_{\mu\nu}\right) = 0.$$
(36)

The particular example given by

$$f(\mathbf{R}) = \mathbf{R} + \alpha \mathbf{R}^2 \tag{37}$$

$$\delta \int (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\sqrt{-g}d^4x = 0,$$

in such a way that the Riemann-squared term can be omitted.

³⁵ More general cases may include terms proportional to $R_{\mu\nu}R^{\mu\nu}$. In principle a term $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ should also be included in the action, but the existence of a topological invariant yields

was studied by many authors [361,184,274]. The equations of motion for the Lagrangian introduced in Eq. (37) in the presence of matter are

$$(1+2\alpha R)R_{\mu\nu} - \frac{1}{2}(R+\alpha R^2)g_{\mu\nu} + 2\alpha(R_{,\mu;\nu} - \Box Rg_{\mu\nu}) = -T_{\mu\nu}.$$
(38)

If we restrict to ultra-relativistic matter ($p = \rho/3$) the 0 – 0 component of this equation in the case of the FLRW geometry yields

$$\rho = \frac{1}{3}\theta^2 + \frac{3\epsilon}{a^2} - 2\alpha\dot{\theta}\left(\dot{\theta} + \frac{2}{3}\theta^2\right) + \frac{18\epsilon^2\alpha}{a^4} + \frac{4\epsilon\alpha}{a^2} + 2\alpha\theta\dot{R},\tag{39}$$

where $R = 2\dot{\theta} + 4\theta^2/3 + 6\epsilon/a^2$, and $\theta = 3\dot{a}/a$. At the point where the bounce occurs, $\theta_b = 0$ and $\dot{\theta}_b > 0$, and Eq. (39) reduces to

$$\rho_b = -2\alpha \dot{\theta}_b^2 + \frac{3\epsilon}{a_b^2} \left(1 + \frac{6\alpha\epsilon}{a_b^2} + \frac{4\alpha}{3} \right). \tag{40}$$

Let us take as an example the case in which $\epsilon = 0$. If we want to have a minimum with positive energy density, it follows from Eq. (40) that $\alpha < 0$. As shown in [361], such a choice for the action of the gravitational field admits solutions in the FLRW framework that allow a regular transition from a contracting to an expanding phase. Although negative values of α remove the initial singularity, it was shown in [361,184] that the solutions with $\alpha < 0$ do not go to the corresponding FLRW solution ($a \propto t^{1/2}$) for large *t*.

A theory that generalizes that defined by Eq. (37), namely

$$f(R) = R + \alpha R^n$$

was studied in [360].³⁶ It was found that the FLRW solution for n = 4/3 and $p = \rho/3$ is regular for all values of *t*, and tends to the radiation solution for large values of *t*. Later, solutions of this theory with dust as a source were found to have similar properties in [195].

Another type of corrections, given by the Lagrangian

$$\mathcal{L} = R + \Lambda + BR^2 + CR^2 \ln |R|, \tag{41}$$

were studied in [196] (with *B* and *C* constants). The quadratic and logarithmic terms are consequences of vacuum polarization [131]. Although this form of the Lagrangian does not eliminate the singularity in the FLRW solutions, addition of particle creation effects through a viscosity term does (see Section 5).³⁷

Stability analysis of the FLRW solution in theories with $\mathcal{L} = f(R)$ was performed in [31], along with necessary and sufficient conditions for the existence of singularities. Eq. (35) in the case of a FLRW geometry in the presence of matter reduces to [238]

$$f''\dot{a}(a^{2}\ddot{a} + a\dot{a}\ddot{a} - 2\dot{a}^{3} - 2\dot{a}\epsilon) + \frac{1}{6}f'a^{3}\ddot{a} + \frac{1}{36}fa^{4} + \frac{1}{18}a^{4}T_{00} = 0.$$
(42)

The argument of the function f is given by

$$R = \frac{6}{a^2} (a\ddot{a} + \dot{a}^2 + \epsilon). \tag{43}$$

Assuming that near the bounce the scale factor can be developed in a power series as

$$a(t) = a_0 + \frac{1}{2}a_1t^2 + \frac{1}{6}a_2t^3 + \cdots,$$
(44)

a necessary condition for the bounce was given [31]:

$$f_0 a_0 + 6a_1 f'_0 \le 0, \tag{45}$$

where $f_0 = f(R_0)$, and $R_0 = -6a_0^{-2}(a_0a_1 + \epsilon)$, and it was assumed that $T_{00} > 0$. In the quadratic case given by Eq. (37), this condition takes the form

$$6\alpha\epsilon^2 - a_0^2\epsilon - 6\alpha a_1^2 a_0^2 < 0. \tag{46}$$

 $^{^{36}}f(R)$ theories with negative/positive powers of *R* were first proposed in [305].

³⁷ A Bianchi I A solution of this theory with and without self-consistent particle production was considered in [197]. It was shown that particle production quickly isotropizes the model.

When $\epsilon = 0$, the condition $\alpha > 0$ is regained, but there are other possibilities when $\epsilon = 1, -1$ [31,53]. In the same vein, but without using a series development, conditions for a bounce in f(R) theories were studied in [95].³⁸ The basic equations are, that follow from Raychaudhuri's equation and the Gauss–Codazzi equation are

$$egin{aligned} & \ddot{a}_b = -rac{
ho_b}{f_b'} + rac{f_b}{f_b'}, \ & R = 6\left(rac{\ddot{a}_b}{a_b} + rac{\epsilon}{a_b^2}
ight). \end{aligned}$$

These equations were used in [95] to analyze a possible bounce in the theories given by $f_1(R) = R^n$, $f_2(R) = R + \alpha R^m$, $f_3(R) = \exp(\lambda R)$. Bounces for $\epsilon = \pm 1$ are possible in the case of f_1 . This case can describe an "almost-FRLW" phase followed by an accelerated phase if n > 1 and n is odd for $\epsilon = -1$ and R > 0. The same happens with n even and n < 0 with R > 0 or 0 < n < 1 with R < 0, where in the second case n can be only rational. For f_2 , closed bounces are allowed for every integer value of m (often together with open bounces). For m rational, closed bounces are not allowed in general for 0 < m < 1. For m rational with even denominator there is no closed bounce for $(m > 1, \alpha < 0)$ and no bounce at all for negative m and α . In the case of f_3 , one of the following two conditions must be satisfied in order to have a bounce: $\lambda > 0$ and $R_b > \ln(2\rho_b)/\lambda$, or $\lambda < 0$ and $R_b < \ln(2\rho_b)/\lambda$.

Some exact solutions have been recently found in [107] for the theory defined by $f(R) = R^{1+\delta}$. For the vacuum case with $\epsilon = 0$, there is bouncing (entirely due to the dynamics of the theory), for $0 < \delta < 1/4$. There are vacuum solutions for $\delta = 1/2$ and $\epsilon \neq 0$, are given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (\kappa - \kappa t^2 \pm t^4) \left(\frac{\mathrm{d}r^2}{1 - \epsilon r^2} + r^2 \mathrm{d}\Omega^2\right).$$

This solution exhibits a bounce for $\kappa > 0$. Bouncing solutions were also obtained for a perfect fluid with $p = (\gamma - 1)\rho$ in the case $\delta = 1/(3\gamma - 1)$.³⁹

We would like to close this section by pointing out that Eq. (42) illustrates the fact that the flatness problem is not a priori a problem in theories other than GR (no definite behavior of $|\Omega - 1|$ with time follows from (42)).

2.2.1. Saturation

An interesting idea was proposed in [238] (see also [245]) to limit the curvature by adding terms in the Lagrangian, following the lines that Born and Infeld [71,252] devised to avoid singularities in electromagnetism. The Born–Infeld Lagrangian, given by

$$\mathcal{L}_{BI} = \beta^2 \left[\sqrt{1 - \frac{\mathscr{H}^2 - \mathscr{E}^2}{\beta^4}} - 1 \right]$$
(47)

is such that the invariant $\mathscr{H}^2 - \mathscr{E}^2$ cannot take values higher than β^4 . The fact that it takes more and more energy to increment the field when it takes values near β^2 is a phenomenon called saturation.⁴⁰ A similar cutoff may be postulated for the curvature tensor when quantum gravitational fluctuations become non-negligible, that is (presumably), when

$$R \approx \ell_{Pl}^{-2} \approx 10^{66} \mathrm{~cm}^{-2}$$

In [238], non-polynomial Lagrangians f(R) were considered such that they reduce to R when $R \ll \ell_{Pl}^{-2}$, and required that $f(R) \rightarrow \text{constant for } R \rightarrow \infty$. This condition is of course not enough to determine the Lagrangian, but a qualitative guess can be made. A typical Lagrangian that fulfills the above given conditions is

$$f(R) = \frac{R}{1 - \ell_{pl}^2 R}.$$
(48)

An approximate solution of the EOM (42) for (48) by a development as a power series of t for $\epsilon = 0$ was built in [141], the solution being non-singular though strongly dependent on the non-linearities of the chosen Lagrangian.

The idea of saturation was subsequently explored in [140], where an explicit nonsingular solution given by

$$a(t) = \sigma \left(1 + \frac{\beta^4 t^2}{\sigma^4}\right)^{1/4},\tag{49}$$

was inserted in Eq. (42), where σ is a small parameter. This expression tends to the radiation-dominated scale factor for $\beta^4 t^2 / \sigma^4 \gg 1$. With this a(t) and using that $R = -3\beta^4 \sigma^4 / a^8$, Eq. (42) can be rewritten as an ordinary linear second-order

³⁸ Bounce solutions were also shown to exist in orthogonal spatially homogeneous Bianchi cosmologies in $f(R) = R^n$ in [188].

³⁹ Cyclic solutions were obtained in the case $\delta = (3\gamma - 4)/(2(7 - 3\gamma))$ for a convenient choice of the integration constants.

⁴⁰ This is analogous to the fact that it takes an infinite amount of energy to accelerate a mass moving with $v \approx c$ in special relativity.

differential equation for f(R). This equation was integrated for all the values of the 3-curvature. The dependence of the resulting f(R) on the chosen form of a(t) was tested in the case $\epsilon = 0$ with that obtained from $a^8(t) = 1 + 2(1 + \alpha)t^2 + t^4$, which has the same asymptotic limit of Eq. (49). The result in this second case is not distinguishable from the first.

A related analysis was carried out in [57], where it was asked that the theory defined by f(R) be asymptotically free (implying that gravity becomes weak at short distances, in such a way that pressure may counteract the gravitational attraction, thus avoiding the singularity), and also ghost-free (so that the bounce is not caused by negative-energy-density matter).⁴¹ The actions studied in [57] that satisfy these requirements were specified by⁴²

$$f(R) = R + \sum_{n=0}^{\infty} c_n R \square^n R,$$
(50)

and can be rewritten in terms of a higher-derivative scalar-tensor action:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\varPhi R + \psi \sum_{1}^{\infty} c_i \Box^i \psi - (\psi(\varPhi - 1) - c_0 \psi^2) \right),$$

from which it follows that $\psi = R$ (from the EOM of Φ). After a conformal transformation and linearization it follows that the EOM for the scalar fields are [57]

$$\psi = 3\Box\phi, \qquad \phi = 2\left(\sum_{1}^{\infty}c_i\Box^i\psi + c_0\psi\right)$$

with $\phi = e^{\phi}$. From these we get

$$\left(1-6\sum_{0}^{\infty}c_{i}\Box^{i+1}\right)\phi\equiv\Gamma(\Box)\phi=0,$$

and the scalar propagator is

$$G(p^2) \propto rac{1}{\Gamma(-p^2)}.$$

It is precisely the function Γ that controls the absence of ghosts and the asymptotic properties of the theory, which was parameterized in [57] as $\Gamma(-p^2) = e^{\gamma(-p^2)}$, with γ analytic. To actually show the existence of bouncing solutions with the properties mentioned above, the scale factor

$$a(t) = a_0 \cosh\left(\sqrt{\frac{\omega}{2}} t\right),$$

was imposed in the equation for G_{00} written in terms of Γ and its derivatives, and compared with the rhs composed of radiation and cosmological constant, thus yielding the following constraints on Γ :

$$\Gamma'(\omega) = \frac{2}{3}\Gamma'(0) - \frac{1}{3\omega},$$
$$2\omega\Gamma'(\omega) - 1 \ge 0$$

(the latter coming from demanding that the bounce be caused by the nonlinearities, and not by the radiation energy density). The authors go on to show that the kinetic operator defined by

$$\gamma(\omega) = k_1 \omega - k_2 \omega^2 + k_4 \omega^4,$$

where k_i are constants, satisfies the constraints and has the correct Newtonian limit. So a bouncing solution that is ghost and asymptotically free exists for the theory defined by Eq. (50),⁴³ although the Lagrangian in the original variable *R* was not exhibited.

2.3. The limiting curvature hypothesis (LCH)

A different proposal to deal with the singularity problem in the higher-order-curvature scenario is to adopt the limiting curvature hypothesis, introduced by Markov [278] as the limiting density hypothesis.⁴⁴ The LHC postulates the existence of a maximum value for the curvature, in such a way that

$$R^2 < \ell_{Pl}^{-4}, \qquad R_{\mu\nu}R^{\mu\nu} < \ell_{Pl}^{-8}, \qquad W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} < \ell_{Pl}^{-8},$$

⁴¹ For the relation between f(R) theories and ghosts, see [104].

⁴² It was shown in [57] that polynomial actions in *R* do not satisfy these requirements.

⁴³ See also [58].

⁴⁴ For boucing solutions that implement this hypothesis through modifications of the EOS, see [358].

etc, and that any geometry must approach a definite nonsingular solution (typically the de Sitter solution) when the limiting curvature is reached. This automatically guarantees that all curvature invariants are finite [279]. A nonsingular higher order theory was constructed in [292] in which every contracting and spatially flat, isotropic universe avoids the big crunch by ending up in a deSitter state enforced by the LCH, for all initial conditions and general matter content (see also [76]).⁴⁵ The action used in [292] was the linear action plus a non-linear term I_2 with the property that

$$I_2(g_{\mu\nu}) = 0 \Leftrightarrow g_{\mu\nu} = g_{\mu\nu}^{DS},$$
(51)

and enforced that $I_2 \rightarrow 0$ for large curvatures using an auxiliary field (see below). In a subsequent paper [75], the method was applied to an isotropic, homogeneous universe, both in vacuum and in the presence of matter. The solutions corresponding to $\epsilon = 1$ display a deSitter bounce. In the case in which matter is present, it is shown that its coupling to gravity is asymptotically free. Later, the model was generalized to include a dilaton field [77], in which case it admits flat bouncing solutions. The starting point is the dilaton gravity action with an added non-linear term (I_2) times a Lagrange multiplier ψ subject to a potential $V(\psi)$:

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} (\nabla \phi)^2 + \frac{1}{\sqrt{12}} \psi \, \mathrm{e}^{\gamma \phi} I_2 + V(\psi) \right). \tag{52}$$

The potential is to be tailored from the EOM and the constraint equations in such a way that I_2 , given by

$$I_2 = \sqrt{4R_{\mu\nu}R^{\mu\nu}},$$

goes to zero for large curvatures. Notice that this form of I_2 satisfies condition (51), so all the curvature invariants are automatically bounded. Restricting to an FLRW metric with k = 0, the EOM are

$$\dot{\psi} = -3H\psi + 6H - \frac{1}{H} \left(\frac{1}{2}\chi^2 + V(\psi) \right), \tag{53}$$

$$\dot{H} = -V'(\psi), \tag{54}$$

$$\dot{\chi} = -3H\chi, \tag{55}$$

with $\chi = \dot{\phi}$, and a prime denotes derivative w.r.t. ψ . An example was given in [77], where

$$V(\psi) = \frac{\psi^2 - \frac{1}{16}\psi^4}{1 + \frac{1}{32}\psi^4}$$
(56)

was chosen. This potential yields the dilaton gravity action at low curvatures, enforces that I_2 go to zero at large curvatures, and enables a bounce. By means of a phase space analysis of Eqs. (53)–(55), it was shown [77] that all the solutions are non-singular, and that some of them display a bounce either with or without the dilaton. In particular, the flat bouncing solutions with a non-zero dilaton interpolate between a contracting dilaton-dominated phase and an expanding FLRW epoch, thus avoiding the graceful exit problem of pre-big-bang cosmology (see below).

One obvious drawback of the LCH is that the non-linear terms are not dictated by first principles: they are chosen in such a way as to render the theory finite.

2.4. Appendix: f(R) and scalar-tensor theories

Higher-order Lagrangians can be related to scalar-tensor gravity (see for instance [392]). Let us start with the function f(R) is given by

$$f(R) = R + \alpha R^2. \tag{57}$$

The EOM that follow from this Lagrangian is

$$2\alpha R_{;\mu\nu} - (1 - 2\alpha R)R_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2}\alpha R^2 + \frac{1}{2}R - 2\alpha \Box R\right) = 0,$$
(58)

the trace of this equation being

$$\Box R - \frac{R}{6\alpha} = 0. \tag{59}$$

It was shown in [392] that this theory is equivalent to the one given by the action

$$S = \int \sqrt{-g} d^4 x \left[(1 + 2\alpha\varphi)R - \alpha\varphi^2 \right].$$
(60)

 $^{^{45}}$ Note that the LFH furnishes in this case a nonsingular universe without bounce.

Varying independently $g_{\mu\nu}$ and φ in the action given in Eq. (60), one obtains

$$(1+2\alpha)\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)+\frac{\alpha}{2}\varphi^{2}g_{\mu\nu}-2\alpha(\varphi_{,\mu;\nu}-\Box\varphi g_{\mu\nu})=0,$$
(61)

and

$$2\alpha(R-\varphi) = 0. \tag{62}$$

In turn, as shown in [417] the conformal transformation

$$\widetilde{g}_{\mu\nu} = (1 + 2\alpha\phi) g_{\mu\nu}, \tag{63}$$

takes this theory to Einstein gravity with a massive scalar field.

Except in the case in which α vanishes (which is precisely the case in general relativity) the second equation yields that the scalar field is nothing but the scalar of curvature. Inserting this result into Eq. (61) one arrives precisely at Eq. (58). The equivalence can be generalized to functions f(R) (see [413]).⁴⁶ Based on this equivalence, the singularity problem in fourth order theories was analyzed in [250] for homogeneous cosmological models with a diagonal metric.

3. Theories with a scalar field

3.1. Scalar field in the presence of a potential

Violations to some of the energy conditions are produced even at the classical level by some scalar field theories. From the singularity theorems discussed in Section 1, we can expect the existence of bouncing solutions in this scenario.⁴⁷ We shall see next examples of avoidance of the singularity in scalar field models that violate some of the energy conditions, as well as theories with nonminimal coupling.

A universe filled with radiation and pressureless matter coupled to a classical conformal massless scalar field was studied in [47]. The coupling was provided by the action

$$S = -\frac{1}{2} \int \left(\psi_{,\alpha}\psi^{,\alpha} + \frac{1}{6}R\psi^2\right)\sqrt{-g} - \int (\mu + f\psi)d\tau,\tag{64}$$

where μ is the mass of the particle, *f* is a coupling constant, and

$$-f\int\psi d\tau = -f\int d^4x \left[\sqrt{-g}\psi\int (-g)^{-1/2}\delta^4(x^{\mu}-x^{\mu}(\tau))d\tau\right],$$

(this interaction was suggested in [47] as a classical analog of the pion–nucleon coupling). Assuming that we have a FLRW universe filled with a uniform distribution of identical μ particles, in the continuum approximation, the field equation for ψ takes the form

$$F_{,\eta,\eta} + \epsilon F = -fN,\tag{65}$$

where $F = a\psi$, η is the conformal time, and $N = na^3 = \text{constant}$ is the number of particles. The calculation of the trace of the total stress-energy tensor from Eq. (64) yields

$$T^{\alpha}_{\alpha} = -\mu n,$$

so we get for the trace of EE

$$a'' + \epsilon a = \frac{4\pi}{3} N\mu, \tag{66}$$

where the prime means derivative w.r.t. conformal time. Finally the Friedmann equation is given by

$$a'^{2} + \epsilon a^{2} = \frac{4\pi}{3} (F'^{2} + \epsilon F^{2} + 2Na\mu + 2NfF + 2B),$$
(67)

where *B* is a constant that gives the amount of radiation. The system composed of Eqs. (65)–(67) was solved in [47] for all values of ϵ , and it was shown that a bounce is possible for the three cases when some relations between the integration constants are fulfilled. However, physical requirements show that only the $\epsilon = +1$ solution can bounce provided $N^2 f^2 > 2B$. A nice feature of this solution is that it satisfies the weak energy condition.

142

⁴⁶ It was later proved that all higher order, scalar-tensor and string actions are conformally equivalent to general relativity with additional scalar fields which have particular (different in each case) self-interaction potentials [32].

⁴⁷ The role of scalar fields in cosmology has been examined for instance in [239].

Another non-singular universe based on a scalar field was presented in [45]. A closed FLRW model was considered, with a conformally coupled scalar field ϕ as matter content, which can be thought as a perfect fluid with comoving velocity defined by

$$u^{\mu} = rac{\phi^{,\mu}}{(\phi_{,\alpha}\phi^{,\alpha})^{1/2}}.$$

In this case, the energy density and the pressure are given by

$$\rho = \frac{1}{2}\dot{\phi}^{2} + \frac{1}{2}\phi^{2}\left[\left(\frac{\dot{a}}{a}\right)^{2} + \frac{1}{a^{2}}\right] + \frac{\dot{a}}{a}\phi\dot{\phi} + V,$$

$$p = \frac{1}{6}\dot{\phi}^{2} + \frac{1}{3}\phi\frac{dV}{d\phi} + \frac{1}{6}\phi^{2}\left[\left(\frac{\dot{a}}{a}\right)^{2} + \frac{1}{a^{2}}\right] + \frac{1}{3}\frac{\dot{a}}{a}\phi\dot{\phi} - V.$$

EE were written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{\rho}{6},$$
$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\gamma - \frac{2}{3}\right)\rho = 0,$$

with $p = (\gamma - 1)\rho$. From these equations we get

$$\frac{\ddot{a}}{a} + \left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}^2 + 1}{a^2}\right) = 0.$$

Introducing the conformal time through $dt = a(\eta)d\eta$, and with the changes of variables u = a'/a, and u = w'/(cw), with $c = 3\gamma/2 - 1$, the solution for $a(\eta)$ is [45]

$$a(\eta) = a_0 [\cos(c\eta + d)]^{1/c},$$

where a_0 , and d are integration constants, which were fixed resorting to the limiting curvature hypothesis (see Section 2.3) along with the imposition of a "prematter" phase (starting from the limiting values), followed by a radiation-dominated era and a matter-domination period afterwards. The constant c is essentially the parameter of the equation of state of the prematter era, and the only constant which is not completely determined in the model. Potential V was then reconstructed in terms of the scale factor (assuming that the EOS changes in the different eras of the universe) and ϕ from $\gamma = 1 + p/\rho$, and the evolution of ϕ was obtained by numerical integration.

More general models, given by solutions of the theory

$$S = \int d^4x \sqrt{-g} \{ F(\phi)R - \partial_\alpha \phi \partial^\alpha \phi - 2V(\phi) \},$$

in which ϕ is nonminimally coupled to gravity through *F*, were studied in [193], where it was shown that there are bouncing solutions, which were later proved to be unstable under linear anisotropic perturbations [3]. A phase-space analysis of the models given by $F(\phi) = \xi \phi^2$ showed the existence of bouncing solutions, under certain restrictions on the constants of the potential $V(\phi) = \alpha \phi^2 + \beta \phi^4 + \Lambda$ [194].

Nonsingular solutions for a scalar field in the presence of a potential were also studied in [11], for theories defined by

$$\mathcal{L} = \frac{1}{2}\omega\dot{\phi}^2 - U(\omega)$$

where ω is determined by $dU/d\omega = \frac{1}{2}\dot{\phi}^2$. The existence of a bounce was shown for a tailored potential given by

$$U(\omega) = \lambda \left(\omega^{-1} + \frac{1-\alpha}{\alpha} \omega^{\alpha/(1-\alpha)} - \frac{1}{\alpha} \right),$$

where λ is a constant with dimensions of energy density, and α is a number parameterising the classes of theories.⁴⁸ The bounce exists for $\alpha < 1/3$, and $\epsilon = +1$. Later, this approach was generalized to Bianchi I cosmologies in [167].

So far we have examined a classical scalar field on a given background. A quantum scalar field $\phi(x)$ in a classical geometry was studied in [286,315] where, inspired by the features of the mechanism of spontaneous symmetry breaking, the authors sought a solution in which the expectation value of ϕ in the fundamental state is given by

$$\langle \mathbf{0}|\phi|\mathbf{0}\rangle = \sqrt{\frac{3}{\lambda}} \frac{f(\eta)}{a(\eta)},\tag{68}$$

⁴⁸ This potential interpolates between $p = \rho$ for $\rho \ll \lambda$, and p < 0 for high densities.

where η is the conformal time of an open Friedmann geometry given by

$$ds^{2} = a^{2}(\eta) \left[d\eta^{2} - d\chi^{2} - \sinh^{2}\chi \left(d\theta^{2} + \sin^{2}\theta d\phi^{2} \right) \right],$$
(69)

and f is a function to be determined (see below). For a massless field the equation of motion for the scale factor reduces to

$$\frac{a''}{a} = 1.$$
(70)

From the Lagrangian

$$\mathcal{L}=rac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-rac{1}{2}\sigma\phi^{4}$$

we obtain the equation of the scalar field ϕ , given by

$$\phi'' + 2\phi'\frac{a}{a} + 2\sigma a^2 \phi^3 = 0.$$
⁽⁷¹⁾

Compatibility of these two equations with the assumption in Eq. (68) yields the relation

$$\sigma = \frac{\lambda}{6}.$$
(72)

For the scale factor as function of the Gaussian time t we obtain

$$a(t) = \sqrt{t^2 - L^2},$$
(73)

where L is a constant and

$$f'' - f + f^3 = 0. (74)$$

By rewriting this equation as a planar autonomous system, it was shown in [286] that the solution f = 0 is unstable, while the solutions $f^2 = 1$ are stable under linear homogeneous perturbations. From the equation for $g_{\mu\nu}$ and specializing for $\mu = \nu = 0$ we obtain the value of the constant *L* in Eq. (73):

$$L^2 = \frac{\kappa}{24\sigma} \tag{75}$$

which represents the minimum allowable value of the scale factor. From standard quantum field theory in curved spacetime,

$$G_{\mu\nu} = -\kappa_{\rm (ren)} T_{\mu\nu},$$

it follows that $E_{|0\rangle} = -\frac{3L^2}{a^4} < 0$, which shows explicitly the expected violation of the weak energy condition that causes the absence of a singularity in this model. Note that the gravitational constant in the vacuum state is renormalized:

$$\frac{1}{\kappa_{(\text{ren})}} = \frac{1}{\kappa} - \frac{\phi^2}{6} = \frac{12\sigma t^2 - \kappa/2}{12\sigma\kappa a^2}$$

It follows that $\kappa_{ren} < 0$ for $t^2 < \frac{\kappa}{24\sigma}$ and $\kappa_{ren} > 0$ for $t^2 > \frac{\kappa}{24\sigma}$, thus showing that a change in the sign of the gravitational constant can be induced by the non-minimal coupling of scalar field with gravity, yielding repulsive gravity.

The phenomenon of repulsive gravity can also be generated at a classical level by means of a non-minimally coupled complex scalar field [367]. The Lagrangian is given by

$$\mathcal{L} = \partial_{\mu}\phi\partial^{\mu}\phi^* - \sigma(\phi^*\phi)^2 - \frac{1}{6}R(\phi^*\phi) + \kappa^{-1}R + \mathcal{L}_m,$$

where σ is the constant that measures the auto-interaction of ϕ , and \mathcal{L} is the matter Lagrangian. The EOM following from this Lagrangian are

$$\Box \phi + 2\sigma \phi^* \phi^2 + \frac{1}{6} R \phi = 0,$$

$$G_{\mu\nu} = -\tilde{\kappa} (\theta_{\mu\nu} + T_{\mu\nu}),$$

where

$$\tilde{\kappa} = \kappa \left(1 - \frac{\kappa}{6} \phi^* \phi \right),$$

$$\theta_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi - g_{\mu\nu} (\partial_\rho \phi^* \partial^\rho \phi - \sigma (\phi^* \phi)^2) + \frac{1}{3} g_{\mu\nu} \Box (\phi^* \phi) - \frac{1}{3} (\phi^* \phi)_{;\mu\nu} \right),$$
(76)

and $T_{\mu\nu}$ is the energy–momentum tensor associated with matter. From Eq. (76) we see that the gravitational constant is renormalized at the classical level by the scalar field. In fact, as shown in [367], for the open FLRW metric⁴⁹ the scalar field has the average and the scalar field of the scalar field.

has three vacuum solutions: $\phi = 0$, and $\phi = \pm \gamma/a(t)$, where γ is a constant. Only the nonzero solutions are stable, and they are also more favorable from the point of view of energy [367]. Since they are inversely proportional to *a*, it may be argued that the scalar field was in a nonzero vacuum in the early universe. Hence,

$$\tilde{\kappa} = \kappa \left[1 - \frac{a_c^2}{a^2} \right]^{-1},$$

where $a_c = (\kappa/12\sigma)^{1/2}$ signals the change of sign of the gravitational interaction. Nonsingular solutions were obtained in [367] for matter given by radiation ($\rho = \epsilon/a^4$):

$$a(t) = \frac{\varpi}{\sqrt{2}}\cosh t,$$

where $\varpi^2 = a_c^2 - \frac{2}{3}\kappa\epsilon$. This case reduces to the case without matter for $\epsilon = 0$.

3.2. Dynamical origin of the geometry

We shall see in this section that a cosmological scenario displaying a bounce arises in an extension of Riemannian geometry called Weyl Integrable Space–Time (WIST) [317].

Let us begin by recalling that one of the central hypotheses of General Relativity is that gravitational processes occur in a Riemannian space–time structure. This means that there exists a metric tensor $g_{\mu\nu}$ and a symmetric connection $\Gamma^{\alpha}{}_{\mu\nu}$ related by

$$g_{\mu\nu;\alpha} \equiv g_{\mu\nu,\alpha} - \Gamma^{\epsilon}_{\alpha\mu} g_{\epsilon\mu} - \Gamma^{\epsilon}_{\alpha\nu} g_{\mu\epsilon} = 0.$$
⁽⁷⁷⁾

In other words, the connection is metric and can be written in terms of the metric tensor as follows

$$\Gamma^{\alpha}_{\ \mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} \equiv \frac{1}{2} g^{\alpha\beta} [g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}].$$
(78)

A direct method to deduce such metricity condition is given by the first order Palatini variation (in which the variation of the metric tensor and of the connection are independent). The starting point is the Hilbert action:

$$\mathscr{S}[g,\Gamma] = \int \sqrt{-g} \mathcal{R}[g,\Gamma] \mathrm{d}^4 x. \tag{79}$$

In a local Euclidean coordinate system,

$$\delta R_{\mu\nu} = \delta \Gamma^{\alpha}_{\ \mu\alpha;\nu} - \delta \Gamma^{\alpha}_{\ \mu\nu;\alpha},\tag{80}$$

where the covariant derivative represented by a semicolon must be taken in the non-perturbed background geometry. From this equation it follows that

$$\delta \mathcal{L} = \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}.$$
(81)

Correspondingly

$$\delta\delta = \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} \right) \delta g^{\mu\nu} + \int \left\{ (\sqrt{-g} g^{\mu\epsilon})_{;\alpha} - \frac{1}{2} (\sqrt{-g} g^{\mu\nu})_{;\nu} \delta^{\epsilon}_{\alpha} - \frac{1}{2} (\sqrt{-g} g^{\nu\epsilon})_{;\nu} \delta^{\mu}_{\alpha} \right\} \delta \Gamma^{\alpha}{}_{\mu\epsilon}.$$
(82)

Hence,

$$(\sqrt{-g}g^{\mu\epsilon})_{;\alpha} - \frac{1}{2}(\sqrt{-g}g^{\mu\nu})_{;\nu}\delta^{\epsilon}_{\alpha} - \frac{1}{2}(\sqrt{-g}g^{\nu\epsilon})_{;\nu}\delta^{\mu}_{\alpha} = 0,$$
(83)

and we obtain

$$(\sqrt{-g}g^{\mu\epsilon})_{;\alpha} = 0.$$
(84)

After some algebra it can be shown that space-time has a Riemannian structure, that is, it obeys the metricity condition,

$$g_{\mu\epsilon;\alpha} = 0. \tag{85}$$

The other equation that follows from the variational principle yields Einstein's equations. The lesson we learn from this calculation is that the structure of the manifold associated to space-time is not given *a* priori, but may depend on the

⁴⁹ This scenario does not work for the closed case.

dynamics. Surely, we should check whether the addition of matter alters this feature. The answer is not unique: it depends crucially on the way matter couples to gravity. There will be no modification to the precedent structure if we adopt the minimal coupling (that is, if the strong equivalence principle is valid). However, when the interaction is non-minimal, the geometrical structure obtained by the Palatini variation is not Riemannian in general. The simplest way to show this is with an example. Let us take the Lagrangian which describes the non-minimal interaction of a scalar field with gravity in the form:

$$L_{\rm int} = \sqrt{-g} R f(\varphi). \tag{86}$$

Following the procedure sketched above we get:

$$\delta S_{\text{int}} = \int \sqrt{-g} f\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \delta g^{\mu\nu} + \int \left\{ (\sqrt{-g}fg^{\mu\epsilon})_{;\alpha} - \frac{1}{2}(\sqrt{-g}fg^{\mu\nu})_{;\nu}\delta^{\epsilon}_{\alpha} - \frac{1}{2}(\sqrt{-g}fg^{\nu\epsilon})_{;\nu}\delta^{\mu}_{\alpha} \right\} \delta \Gamma^{\alpha}{}_{\mu\epsilon}, \tag{87}$$

and it follows that

$$\left\{\sqrt{-g}f(\varphi)g^{\mu\nu}\right\}_{;\epsilon} = 0.$$
(88)

This equation shows that the covariant derivative of the metric tensor is not zero but

$$g_{\mu\nu;\alpha} = Q_{\mu\nu\alpha},\tag{89}$$

where $Q_{\mu\nu\lambda} = -(\ln f)_{,\lambda}g_{\mu\nu}$. Taking the cyclic permutation of Eq. (89) yields

$$\Gamma^{\lambda}{}_{\mu\alpha} = { \lambda \atop \mu\alpha} - \frac{1}{2} [Q_{\mu}{}^{\lambda}{}_{\alpha} + Q^{\lambda}{}_{\alpha\mu} - Q_{\alpha\mu}{}^{\lambda}].$$
⁽⁹⁰⁾

The equation

$$g_{\mu\nu;\alpha} = -(\ln f)_{,\lambda} g_{\mu\nu} \tag{91}$$

shows that the structure generated by the Lagrangian (86) using the Palatini variation is not Riemannian but, as we shall see in the next section, a special case of Weyl geometry.

3.2.1. WIST (Weyl Integrable Space Time)

A Weyl geometry is defined by the relation [415]

$$g_{\mu\nu;\alpha} = \varphi_{\alpha}g_{\mu\nu}. \tag{92}$$

This equation implies that there is a variation of the length ℓ_0 of any vector under parallel transport, given by

$$\Delta \ell = \ell_0 \varphi_\mu \Delta x^\mu. \tag{93}$$

This property has the undesirable consequence that the measure of length depends on the previous history of the measurement apparatus, as pointed out by Einstein in the beginning of the past century in a criticism against Weyl's proposal for the geometrization of the electromagnetic field [334]. Einsteins remark led to the abandonment of this type of geometry. However, there is just one particular case in which this problem disappears: the so-called Weyl integrable spacetime (WIST). By definition, a WIST is a particular Weyl spacetime in which the vector W_{μ} is irrotational:

$$\varphi_{\mu} \equiv \partial_{\mu}\varphi$$

It follows that in a closed trajectory

$$\oint \Delta \ell = 0, \tag{94}$$

which solves the critic raised by Einstein. From the definition given in Eq. (92) it follows that the associated connection is given by

$$C^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \frac{1}{2} \left(\varphi_{\mu} \delta^{\alpha}_{\nu} + \varphi_{\mu} \delta^{\alpha}_{\nu} - \varphi^{\alpha} g_{\mu\nu} \right).$$
(95)

Using this equation we can write the contracted curvature tensor $R_{\mu\nu}^{(W)}$ in terms of the tensor $R_{\mu\nu}$ of the associated Riemann space constructed with the Christoffel symbols $\left\{ \begin{array}{l} \alpha \\ \mu\nu \end{array} \right\}$. We obtain

$$R_{\mu\nu}^{(W)} = R_{\mu\nu} - \varphi_{,\mu;\nu} - \frac{1}{2}\varphi_{,\mu}\varphi_{,\nu} + \frac{1}{2}\varphi_{,\lambda}\varphi^{,\lambda}g_{\mu\nu} - \frac{1}{2}\Box\varphi g_{\mu\nu}$$
(96)

where the covariant derivatives are taken in the associated Riemannian geometry and \Box is the d'Alembertian in the Riemannian geometry. Thus, for the curvature scalar,

$$R^{(W)} = R - 3\Box \varphi + \frac{3}{2} \varphi_{,\lambda} \varphi^{,\lambda}$$
(97)

in which *R* is the curvature scalar of the associated Riemannian spacetime.

The expressions in Eqs. (96) and (97) are very similar to those obtained by a conformal mapping of a Riemannian geometry as shown in Section 3.4.

3.2.2. WIST duality: The Weyl map

A Weyl integral spacetime is determined by both a metric tensor and a scalar field. In [415], Weyl introduced a generalization of the conformal mapping, which he called a gauge transformation, given by

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{\chi} g_{\mu\nu}, \qquad \varphi \to \tilde{\varphi} = \varphi + \chi, \tag{98}$$

in which χ is an arbitrary function. Under such transformations the affine connection and the curvature and Ricci tensors are invariant:

$$\begin{split} & C^{\alpha}{}_{\mu\nu} = C^{\alpha}{}_{\mu\nu}, \\ & \tilde{R}^{(W)\alpha}_{\beta\mu\nu} = R^{(W)\,\alpha}{}_{\beta\mu\nu}, \\ & \tilde{R}^{(W)}_{\mu\nu} = R^{(W)}_{\mu\nu}. \end{split}$$

Note however that this is not the case for the scalar of curvature, which changes as

 $\tilde{R}^{(W)} = \mathrm{e}^{-\chi} R^{(W)}.$

This property has been used to construct gauge-invariant theories, as we shall see next.

3.2.3. Invariant action in WIST

From the behavior of the geometric quantities under a Weyl map, it is not difficult to write an action that is invariant under the transformation given by Eq. (98):

$$S_W = \int \sqrt{-g} \, \mathrm{e}^{-\varphi} \, \mathrm{R}^{(W)}. \tag{99}$$

This Lagrangian can be rewritten in terms of the associated Riemannian quantities as follows:

$$S_{\rm W} = \int \sqrt{-g} e^{-\varphi} \left(R - 3\Box \,\varphi + \frac{3}{2} \,\varphi_{,\lambda} \varphi^{,\lambda} \right). \tag{100}$$

After some algebra, we arrive (up to a total divergence) at the result

$$S_W = \int \sqrt{-g} e^{-\varphi} \left(R - \frac{3}{2} \varphi_{,\mu} \varphi^{,\mu} \right).$$
(101)

Note that the kinematical term of the scalar field for the scalar field appears with the "wrong" sign. This can be interpreted as a ghost field term hidden in the WIST structure.

3.2.4. A particular case of WIST duality

Let us go one step further and add to the above Lagrangian a kinematical term:

$$S_{\mathcal{K}} = \int \sqrt{-g} e^{-\varphi} \varphi_{,\mu} \varphi^{,\mu}.$$
(102)

If we restrict to the case in which χ (given in Eq. (98)) is a functional of φ , it follows that the complete action

$$S = \int \sqrt{-g} \, \mathrm{e}^{-\varphi} (R^{(W)} + \beta \varphi_{,\mu} \varphi^{,\mu}) \tag{103}$$

is invariant under the restricted map

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-2\varphi} g_{\mu\nu}, \tag{104}$$
$$\varphi \to \tilde{\varphi} = -\varphi,$$

which is a special case of the general transformation (98). In terms of Riemann variables,

$$S = \int \sqrt{-g} \, \mathrm{e}^{-\varphi} \left[R + \left(\beta - \frac{3}{2} \right) \varphi_{,\mu} \varphi^{,\mu} \right]. \tag{105}$$

There are three invariants of dimension $(\text{length})^2$ that can be constructed with the independent quantities of a WIST geometry: $R^{(W)}$, $\varphi^{\alpha} \varphi_{\alpha}$, and $\varphi^{\alpha}_{;\alpha}$, where $\varphi_{\alpha} \equiv \varphi_{,\alpha}$. Now, since the covariant derivative ";" in the WIST spacetime can be written in terms of the Riemann covariant derivative (denoted by "||") as

$$arphi^lpha_{\,;lpha}=arphi^lpha_{\,\parallellpha}-2arphi^{,lpha}arphi_{,lpha},$$

the three invariants reduce to two. The most general action can then be written as

$$S = \int \sqrt{-g} \left[R^{(W)} + \xi \varphi^{\alpha}_{;\alpha} \right], \tag{106}$$

where ξ is a constant. Independent variation of the metric tensor and the WIST field φ yields

$$\Box \varphi = 0,$$

(the operator \Box is calculated in Riemannian spacetime) and

$$R^{(W)}{}_{\mu\nu} - \frac{1}{2} R^{(W)} g_{\mu\nu} + \varphi_{,\mu;\nu} - 2(\xi - 1)\varphi_{,\mu}\varphi_{,\nu} + \left(\xi - \frac{1}{2}\right)g_{\mu\nu}\varphi_{,\alpha}\varphi^{,\alpha} = 0.$$
(108)

(107)

This equation can be rewritten exclusively in terms of the associated Riemannian structure

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \lambda^2 \varphi_{,\mu} \varphi_{,\nu} + \frac{\lambda^2}{2} \varphi_{,\alpha} \varphi^{,\alpha} g_{\mu\nu} = 0,$$
(109)

where

$$\lambda^2 = \frac{1}{2} \left(4\xi - 3 \right). \tag{110}$$

3.2.5. A nonsingular cosmological model in WIST

Let us now show how a nonsingular cosmological scenario in the WIST framework can be constructed, following [317]. We shall work with the standard form of the FLRW metric:

$$ds^{2} = dt^{2} - a^{2}(t) \left[\frac{dr^{2}}{1 - \epsilon r^{2}} + r^{2}(d\theta^{2} + \sin^{2}\theta d\varphi^{2}) \right].$$
(111)

As in the case of a standard scalar field, the WIST configuration can be represented by a perfect fluid, so that Eq. (109) becomes Einsteins equation for a perfect fluid with $v^{\mu} = \delta^{\mu}_{0}$, energy density ρ_{φ} and pressure p_{φ} , given by

$$\rho_{\varphi} = p_{\varphi} = -\frac{1}{2} \lambda^2 \dot{\varphi}^2. \tag{112}$$

In this interpretation, the WIST structure is equivalent to a Riemannian geometry, satisfying the equations of General Relativity with a perfect fluid having negative energy density as a source. The gauge vector φ_{λ} for this geometry becomes

$$\varphi_{\nu} = \partial_{\lambda}\varphi(t) = \dot{\varphi}\,\delta_{\lambda}^{0},\tag{113}$$

where the dot denotes differentiation with respect to the time variable. Use of Eq. (107) yields a first integral for the function $\varphi(t)$:

$$\dot{\varphi} = \gamma a^{-3},\tag{114}$$

where $\gamma = \text{constant}$. In turn, EE (109) for the Friedman scale factor a(t) are

$$\dot{a}^2 + \epsilon + \frac{\lambda^2}{6} (\dot{\varphi}a)^2 = 0,$$
 (115)

$$2a\ddot{a} + \dot{a}^2 + \epsilon - \frac{\lambda^2}{2} (\dot{\varphi}a)^2 = 0,$$
(116)

where ϵ is the 3-curvature parameter of the FLRW geometry. From Eq. (115) we see that $\epsilon = -1$. Combining Eqs. (114) and (115) we get the fundamental dynamical equation

$$\dot{a}^2 = 1 - \left[\frac{a_0}{a}\right]^4,\tag{117}$$

with $a_0 = [\gamma^2 \lambda^2/6]^{1/4}$. Before entering into the details of the solution of the system of structural and dynamical equations (115) and (114), let us comment some of the consequences of this cosmological model and list some interesting results.

Features of the model

An immediate consequence of Eq. (117) is that the scale factor a(t) cannot attain values smaller than a_0 . Let us consider a time reversal operation and run backwards into the past of cosmic evolution. As the cosmic radius a(t) decreases, the temperature of the material medium grows. In Hot Big Bang models such an increment is unlimited; in the present theory, on the other hand, there is an epoch of greatest condensation in the vicinity of the minimum radius a_0 . Close to this period, there occurs a continuous "phase transition" in the geometrical background: a Weyl structure is activated, according to Eq. (114): the Universe attains the minimum radius a_0 at (t = 0), and consequently an unbounded growth of temperature is inhibited. Notice that since the Universe had this infinite collapsing era to become homogeneous, in the present scenario the horizon problem of standard cosmology does not arise.

For very large times, the scale factor behaves as $a \sim t$. Thus, asymptotically, the geometrical configuration assumes a Riemannian character (since $\dot{\varphi} \rightarrow t$) in the form of a flat Minkowski space (in Milne's coordinate system). Consequently,

in the present model the evolution of the universe may be started by a primordial instability of Minkowski spacetime at the remote past, due to Weyl perturbations of the Riemann structure through Eq. (91). In order to prescribe the behavior of these perturbations, knowledge of the time dependence of the gauge vector φ_{λ} is required. Since the WIST function $\dot{\varphi}$ has a maximum at t = 0, the largest deviation of the Riemannian configuration corresponds to the epoch of greatest contraction near to the value a_0 .

Stability of the solution

Among the difficult questions concerning bouncing Universes, one may count the problem of their survival with respect to eventual metric perturbations (see Section 11). We shall show that during the stage of greatest condensation the WIST model of the Universe is stable. Applying the homogeneous perturbations

$$\begin{aligned} \varphi &\to \varphi + \delta \varphi, \\ a &\to a + \delta a, \end{aligned}$$

to Eqs. (114) and (115), one obtains

$$\delta \dot{\varphi} = -\frac{3\gamma}{a^4} \, \delta a,$$

 $\delta \dot{a} \sim 2 \frac{a_0^4}{a^5 \dot{a}} \, \delta a.$

Hence,

$$\delta \dot{\varphi} = -rac{9 \dot{a}}{\gamma \lambda^2} \; a \delta \dot{a},$$

 $rac{\delta \dot{a}}{\delta a} \sim a^{-3} [a^4 - a_0^4]^{-rac{1}{2}}$

Far from a_0 (i.e., for large t) we have $a \gg a_0$; then,

$$rac{\delta \dot{a}}{\delta a} \sim a^{-5}, \ \mathrm{d} a \sim \mathrm{d} t,$$

so with $(\delta a)_i$ being the initial spectrum of perturbations, one obtains

 $\delta a \sim (\delta a)_i \exp[a^{-4}].$

The solutions of the system equations (114) and (115) are therefore stable against metric perturbations in the course of the infinite collapsing phase.

The exact solution

No closed solution can be obtained in terms of the cosmological time, so it is convenient to move to conformal time η , in which case the solution is easily shown to be

$$a(\eta) = a_0 \sqrt{\cosh 2(\eta - \eta_0)},\tag{118}$$

where η_0 is an integration constant. The following qualitative plot (Fig. 1) shows the difference between this bouncing solution and the radiation-dominated model in standard cosmology. The scale factor has a minimum for $a = a_0$, which corresponds to $\eta = \eta_0$. Thus the Universe had a collapsing era for $\eta < \eta_0$, attained its minimum dimension at $\eta = \eta_0$, and thereafter initiated an expanding era. Both the collapse and the expansion run adiabatically, i.e., at a very slow pace.

The correlate behavior of the Hubble expansion parameter $H \equiv (\dot{a}/a)$ helps to understand the model (Fig. 2). Indeed, the Hubble parameter *H* is a smooth function of conformal time η and does not diverge at the origin of the expanding era; quite on the contrary, it vanishes at $\eta = \eta_0$. The corresponding evolution of the Cosmos may be outlined as follows: the Universe stays for a long period in a phase of slow adiabatic contraction, until *H* attains its minimum value. Then an abrupt transition occurs: a fast compression turns into a fast expansion up to the maximum of *H*, and afterwards the expansion proceeds in an adiabatic slow pace again. While this image supplies a picture of the behavior of an Universe driven by $\varphi(t)$, it is however incomplete, due to the fact that the production of large amounts of matter and entropy has been neglected. This topic will be discussed in Section 3.2.9).

3.2.6. The WIST function $\varphi(t)$: Structural transitions

According to the basic conception of the scenario presented above, the WIST function $\varphi(t)$ governs the cosmic evolution. Taking into account the solution Eq. (118) for the scale factor a(t), the first integral equation (114) yields for $\varphi(t)$ the expression

$$\varphi = \frac{\gamma}{2a_0^2} \arccos\left[\frac{a_0}{a}\right]^2.$$
(119)



Fig. 1. The qualitative plot shows (in conformal time) the scale factor for the bouncing model given by Eq. (118), and the scale factor for radiation in the SCM, $a \propto \eta$.



Fig. 2. Plot of the Hubble parameter in conformal time for $a_0 = 1$.

The behavior of $\varphi(t)$ is qualitatively portrayed in Fig. 3, along with $\dot{\varphi}$. Note that when $a \to \pm \infty$ (i.e., for large times), $\varphi \to \pm \gamma \pi/4a_0^2 = \text{constant}$, which is consistent with the assumption that the Universe originated from a Minkowskian "nothing" state. The behavior of the time derivative $\dot{\varphi} = \gamma/a^3$, which appears in Eq. (112) of the energy density ρ_{φ} of the "stiff matter" state associated to the WIST field is also shown in Fig. 3. Since this function has a strong peak in the neighborhood of the minimum radius a_0 , the greatest deviation from the Riemannian configuration happens at this point. In this sense, a sort of "structural phase transition" takes place when the Universe approaches its maximally condensed state. The increase of the (negative) energy of the WIST "fluid" precludes the collapse to a singularity, reversing the cosmic evolution into an expansion. Note that the "kinky" aspect of the behavior of the WIST function $\varphi(t)$ in Fig. 3 suggests a similarity between the Weyl structural transition described above and the propagation of instantons in Euclideanized models of quantum creation (see Eq. (117)).

3.2.7. WISTons and anti-WISTons: On the geometrization of instantons

In the derivation of the solution of the WIST structural function $\varphi(t)$ (given by Eq. (119)), no attention was paid to the sign of the constant γ . Since the only information we have about γ is that $\gamma^2 = 6a_0^4/\lambda^2$, according to Eqs. (114) and (117), γ can be either positive or negative:

$$\gamma^{(\pm)}=\pm\sqrt{6}rac{a_0^2}{|\lambda|}.$$

Hence, Eqs. (114) and (119) actually yield two equations, as follows:

$$\varphi^{(\pm)} = \varphi_0^{(\pm)} \arccos\left[\frac{a_0}{a}\right]^2,$$

$$\dot{\varphi}^{(\pm)} = \frac{\gamma^{(\pm)}}{a^3},$$
(120)



Fig. 3. Plot of φ (full line) and $\dot{\varphi}$ (dotted line) in conformal time for $a_0 = 1$.

in which $\varphi_0^{(\pm)} = \gamma^{(\pm)}/2a_0^2 = \pm \sqrt{3/2} |\gamma|^{-1}$. Thus the amplitude of the solutions $\varphi^{(\pm)}$ depends exclusively on the dimensionless parameter ξ (see Eq. (110)). The plot of the WIST functions $\varphi^{(-)}(t)$ and $\dot{\varphi}^{(-)}(t)$ is given by the mirror image of Fig. 3 with respect to the horizontal axis. Note, however, that the energy density ρ_{φ} of the "stiff matter" state associated with the WIST field $\varphi(t)$ is the same in both cases, since from Eqs. (112) and (114) we have

$$\rho_{\varphi} = -\frac{\lambda^2}{2} \dot{\varphi}^2 = -3 \left[\frac{a_0^4}{a^6} \right]. \tag{122}$$

Thus, in spite of the fact that the pairs of WIST functions $(\varphi^{(+)}, \dot{\varphi}^{(+)})$ and $(\varphi^{(-)}, \dot{\varphi}^{(-)})$ have different characteristics, they induce the same type of cosmological evolution. Their only distinction, in fact, is connected to length variations, since according to Eq. (93) one now has $\Delta L^{(\pm)} = L\dot{\varphi}^{(\pm)}\Delta t$.

It is interesting to observe that the system is invariant with respect to the time reversal operation $t \rightarrow (-t)$ if $\varphi^{(+)}$ is concurrently mapped into $\varphi^{(-)}$ and reciprocally. In this sense, the WIST instanton-like functions $\varphi^{(+)}$ and $\varphi^{(-)}$ may be called "WISTon" and "anti-WISTon" solutions, respectively, since an anti-WISTon may be described as a WISTon running backwards in time. According to Eq. (107), WISTons are defined up to an additive constant.

A closer inspection of the equations governing the behavior of $\varphi(t)$ reveals an instanton-like behavior typical of nonlinear theories of self-interacting scalar fields. Of course, the root of such nonlinearity is the fact that $\varphi(t)$ is taken as the actual source of the curvature of the metric structure, which in turn modifies the D'Alembertian operator \Box due to the introduction of φ -dependent terms. A direct way to clarify this issue is to make explicit, by means of a change of variables, the hidden nonlinearity of the system of equations of motion involving the scale factor a(t) and the WIST function $\varphi(t)$. Define the new variable $s(t) = \dot{\varphi}(t)$. Using Eqs. (107) and (114), we have

$$\begin{cases} \dot{s} + 3\gamma a^{-4} \dot{a} = 0, \\ a^3 - \gamma s^{-1} = 0. \end{cases}$$
(123)

Taking s(t) to represent a generalized coordinate associated with a one-particle dynamical system yields the conservation equation

$$\frac{1}{2}\dot{s}^2 + V(s) = 0, \tag{124}$$

in which the associate potential V(s) is given by

$$V(s) = \frac{9}{2\gamma^2} \left[a_0^4 s^4 - \gamma^{\frac{4}{3}} s^{\frac{8}{3}} \right]$$

= $\frac{3\lambda^2}{4} \left[s^4 - b^2 s^{\frac{8}{3}} \right],$ (125)

with $b^2 = 6\lambda^{-2}\gamma^{2/3}$. Thus the evolution of field *s* is equivalent to a unit mass particle moving in a potential with vanishing total energy. Due to the nonlinear character of this potential, the instanton-like aspect of functions $\varphi^{(\pm)}(t)$ is not surprising. Fig. 4 shows the behavior of V(s). The potential vanishes at s = 0 and at $s_B^{(\pm)} = \gamma^{(\pm)}a_0^{-3}$ its extrema are at s = 0, and at $s_m^{(+)} = (2/3)^{3/4}\gamma^{(\pm)}a_0^{-3}$ (which are minima). However, the system cannot remain at the stable states $V(s_m^{(\pm)}) = \left(-\frac{2}{3}\right)\gamma^2a_0^{-8}$, since in this case $\dot{s} \neq 0$; this in turn implies, of course, a nontrivial, evolving cosmic configuration. This nonlinear scheme provides a succinct picture of the evolution of the Universe: its development is initiated at s = 0 (which corresponds to Minkowski space time at $t \to -\infty$), attains its minimum radius $a(t = 0) = a_0$ at either $s_B^{(+)}$ or $s_B^{(-)}$ and returns back to



Fig. 4. Qualitative plot of V(s).

s = 0 (which now corresponds to a Minkowski spacetime at $t \to +\infty$). According to whether the system proceeds along the right or the left branches (i.e., from s = 0 to $s_B^{(+)}$ or $s_B^{(-)}$) of the figure, the cosmic evolution is driven by a WISTon or an anti-WISTon, respectively.

The appearance of instanton-like configurations is a direct consequence of the fundamental dynamical equation (117), in combination with the "structural" equation (114) which prescribes the degree of "Weylization" of space time.

3.2.8. Weylization

We shall see next that the "structural transitions" discussed above are equivalent to a quantum tunneling process in models of quantum creation from "nothing". Consider a generic Einstein equation for a Friedman scale factor,

$$\dot{a}^2 = -\epsilon + \frac{1}{3}\,\rho a^2.\tag{126}$$

It was shown in [108] that a semiclassical description of a quantum tunneling process is given by the bounce solutions of Euclideanized field equations, i.e., of field equations in which the time parameter t is changed into (-it). Applying such an Euclideanization procedure to Eq. (126), one obtains

$$\dot{a}^2 = +\epsilon - \frac{1}{3}\,\rho a^2.\tag{127}$$

In the case of an $\epsilon = +1$ universe driven by a (positive) cosmological constant $\Lambda = 3\varsigma^2$ this approach was used in [408] to obtain, instead of the classical de Sitter solution, namely

$$a(t) = \frac{1}{\varsigma} \cosh(\varsigma t),$$

the solution

$$a_E(t) = \left(\frac{1}{\varsigma}\right)\cos(\varsigma t),\tag{128}$$

corresponding to a de Sitter instanton – a "kink" configuration – propagating with negative classical energy, which bounces at the classical turning point $a = a_0 = (1/\varsigma)$ interpreted as representing the tunnelling to classical de Sitter space from "nothing".

Now consider Eq. (126) in the case of a closed Universe driven by the energy density $\rho = 3[a_0^4/a^6]$. The euclideanized version of Eq. (127) gives

$$\dot{a}^2 = 1 - \left[\frac{a_0^4}{a^4}\right].$$

But this is precisely the fundamental dynamical equation (117) of the WIST cosmological scenario. In this way, an equivalence is established between the Euclideanization of a closed Universe model driven by a positive energy density and a "structural transition" to a Weyl configuration which results in an *open* Universe model driven by a "stiff matter" state of negative energy. Just as in models of quantum creation the propagation of an instanton is seen to represent the tunneling of the Universe from a primordial quantum "nothing" state, in the present scenario the propagation of a WISTon (i.e., a deviation of the Riemannian structure) is tantamount to the development of the Universe from a primordial empty Minkowski space.

It has been argued that solutions obtained through Euclideanization are in fact non-realistic, since they are to be interpreted as instantons, field configurations which tunnel across a classically forbidden region. Other authors endorse



Fig. 5. Plot of *a* and φ for $k_0 = 1/7$ and $a_0 = 0.93$, values chosen by imposing that the solution in Eq. (132) enters the radiation era for $t \approx 10^{-8}$ s.

the view that such solutions correspond to an actual primordial phase of the cosmic evolution in which the basic Lorentzian nature of spacetime is changed into an Euclidean one. According to the present model, a different interpretation may be ascribed to these solutions, since an enlargement of the spacetime structure to a Weyl configuration – in which the geometry is characterized by the pair $(g_{\mu\nu}, \varphi_{\lambda})$ of fundamental variables – supplies, at least in a particular case, the same basic behavior. It then becomes possible to reconcile the opposing interpretations of an "abstract soliton configuration" [136] and of a truly observable Euclidean cosmic phase [207]. The WIST solution is observable, whereas its basic nature is always Lorentzian. It is the Riemannian character of spacetime structure that results altered; allegorically, the choice is no longer Euclid or Lorentz, but rather Riemann or Weyl.

3.2.9. Solution with matter generation

We have mentioned above that the model must be improved by taking into account matter creation. A non-singular solution in WIST that incorporates the effect of the creation of matter on the geometry was studied in [364]. Friedmann equation in conformal time is given by

$$a'^{2} - a^{2} = -\frac{\lambda^{2}}{6}(\varphi'a)^{2} + \frac{a^{4}}{3}\rho_{m},$$
(129)

while the second EE is

$$-3\left(2\frac{a''}{a^3} - \frac{a'^2}{a^4} - \frac{1}{a^2}\right) = \rho_m + 3\rho_\varphi.$$
(130)

The conservation of the stress-energy tensor in the case of ultra-relativistic matter is

$$(a^4\rho_m)' + \frac{1}{a^2}(a^6\rho_{\varphi})' = 0.$$
(131)

A particular solution to these equations that describes creation of relativistic matter only around the bounce, and enters a radiation phase with a constant scalar field in a short time is given by the expression [364]

$$a(\eta) = \beta \sqrt{\cosh(2\eta) + k_0 \sinh(2\eta) - 2k_0 (\tanh \eta + 1)},$$
(132)

with $\beta = a_0/\sqrt{1 - k_0}$, and $0 < k_0 < 1/7$. The dependence of φ on η can be obtained from Eqs. (129) and (130). An asymmetry is to be expected both in the scale factor and in φ , since the evolution of this universe starts from the vacuum and enters a radiation dominated epoch. This is pictured in Fig. 5. Notice that since the scalar field tends rapidly to a constant value, the production of matter (controlled by φ' , see Eq. (131)) stops soon, and the model enters a radiation phase without the need of a potential. In this sense, this solution describes a hot bounce, as opposed to cold bouncing solutions, which do not enter the radiation era unless they are heated up [185]. Another nice feature of this solution is that the scalar field (formally equivalent to the dilaton of string theory) goes automatically to a constant value for $\eta \to \infty$, in such a way that the solution could be taken as the leading order of a perturbative development (as is the case in string theory). Again, no potential was needed in order to display this feature.

3.3. Scalar–tensor theories

Scalar-tensor theories are a generalization of the Brans-Dicke Lagrangian [80], in which the constant appearing in the kinetic term of the scalar field ϕ becomes a function of ϕ . Among the possible Lagrangians to describe these theories, one possibility is [385]

$$\mathcal{L} = -f(\phi)R + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} + 16\pi\mathcal{L}_{\text{matter}},$$
(133)

where the scalar field ϕ couples non-minimally with the curvature through $f(\phi)$. With the redefinition $\varphi = f(\phi)$, the Lagrangian becomes

$$\mathcal{L} = -\varphi R + \frac{\omega(\varphi)}{\varphi} \varphi_{,\mu} \varphi^{,\mu} + 16\pi \mathcal{L}_{\text{matter}},$$
(134)

with $\omega(\varphi) = \frac{1}{2}f/f_{\varphi}^2$ and $f_{\varphi} \equiv df/d\varphi$. Brans–Dicke theory is a special case of this Lagrangian, $f(\phi) \propto \phi^2$ which entails $\omega = \text{const.}$ This Lagrangian also describes the gravity-dilaton sector of low-energy string theory for $\omega = -1$ [121]. The differences between the two Lagrangians have been analyzed in [256]. Following the results of the discussion presented there, we shall use Eq. (134) as the definition of scalar–tensor theories.

The equations of motion corresponding to Eq. (134) are

$$R_{\mu\nu} = -\frac{1}{\varphi} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) - \frac{\omega(\varphi)}{\varphi^2} \varphi_{,\mu} \varphi_{,\nu} - \frac{1}{\varphi} \varphi_{,\mu;\nu} - \frac{1}{2\varphi} g_{\mu\nu} \Box \varphi, \tag{135}$$

$$[3+2\omega(\varphi)]\Box\varphi = T - \omega_{\varphi}\varphi_{,\mu}\varphi^{,\mu}.$$
(136)

Eq. (135) suggests that it may be possible to find solutions in which matter satisfies SEC, but the whole rhs is such that $R_{\mu\nu}v^{\mu}v^{\nu} \ge 0.5^{0}$ This implies, via the singularity theorem given in Section 1.2 that nonsingular solutions may exist in scalar-tensor theories. Using Eq. (135), the inequality $R_{\mu\nu}v^{\mu}v^{\nu} \ge 0$ translates for the flat FLRW case and EOS $p = \lambda \rho$ to

$$-\frac{1}{\varphi}(1+3\lambda)\rho\frac{\omega+2}{2\omega+3} - \frac{\dot{\varphi}^2}{\varphi} \left(\frac{\omega}{\varphi} - \frac{\omega'}{2(2+3\omega)}\right) - \frac{\ddot{\varphi}}{\varphi} \ge 0.$$
(137)

Solutions satisfying this constraint, and hence exhibiting a bounce, have been presented in [34], for $\epsilon = 0$ in the cases of vacuum and radiation (for which T = 0, see rhs of Eq. (136)).⁵¹ With these restrictions, Eq. (136) written in conformal time takes the form

$$\varphi'' + \frac{2a'}{a}\varphi' = -\frac{\varphi^2\omega_\varphi}{3+2\omega},\tag{138}$$

which integrates to

$$\varphi'a^2 = \frac{\sqrt{3}A}{\sqrt{2\omega+3}},\tag{139}$$

where A is a constant. Introducing the variable $y = \varphi a^2$ and using Eq. (139), the Friedmann equation takes the form

$$y'^2 = 4\Gamma y + A^2,$$
 (140)

($\Gamma \ge 0$ is a constant coming from energy conservation) yielding for $y(\eta)$,

$$y(\eta) = A(\eta + \eta_0) \tag{141}$$

in the case of vacuum, and

$$y(\eta) = \Gamma(\eta + \eta_0)^2 - \frac{A^2}{4\Gamma}$$
 (142)

in the case of radiation. Dividing now Eq. (139) by $y = \varphi a^2$ we obtain

$$\int \frac{\sqrt{2\omega(\varphi) + 3}}{\varphi} \, \mathrm{d}\varphi = \sqrt{3}A \int \frac{\mathrm{d}\eta}{y(\eta)}.$$
(143)

⁵⁰ The same happens in some wormhole configurations in Brans-Dicke theory. See [13].

⁵¹ A shadow of doubt has been cast on these results in [229], where it was shown that gravitons would still see a singularity, even if the rest of matter does not.

If this equation is such that it yields $\varphi = \varphi(\eta)$, we could obtain $a(\eta)$ from $y = \varphi a^2$. To integrate Eq. (143), we need to specify the function $\omega(\varphi)$. The choice in [34] was

$$2\omega(\varphi) + 3 = 2\beta \left(1 - \frac{\varphi}{\varphi_c}\right)^{-\alpha},\tag{144}$$

where α , $\beta > 0$ and φ_c are constants. With this choice of ω , Eq. (143) can be solved for $\varphi(\eta)$ in the cases $\alpha = 0$ (which corresponds to Brans–Dicke theory), $\alpha = 1$ and $\beta = -\frac{1}{2}$ (which defines a theory introduced by Barker [30]), and $\alpha = 2$. The latter was studied in [34]. The solutions for the vacuum case are given by

$$a(\eta)^{2} = \frac{A(\eta + \eta_{0})(1 + (\eta + \eta_{0})^{\lambda})}{\varphi_{c}(\eta + \eta_{0})^{\lambda}}$$
(145)

$$\varphi(\eta) = \frac{\varphi_c(\eta + \eta_0)^{\lambda}}{1 + (\eta + \eta_0)^{\lambda}},\tag{146}$$

with $\lambda = \sqrt{3/2\beta}$. These solutions were shown to be nonsingular for $\beta < 3/2$. Hence the radiation solutions (which approach those for the vacuum for $\eta \to 0$ [34]) are also nonsingular. All the solutions for $\alpha = 2$ approach the FLRW radiation regime at late times because φ tends to a constant, and then $\omega(\varphi) \to \infty$, but in order to be in agreement with solar system experiments, α must be greater than 1/2 [34].

The case of stiff matter (defined by $\rho = p$) sourcing the scalar field was studied in [289]. Since the density of a barotropic fluid ($p = (\gamma - 1)\rho$) evolves as $\rho \propto a^{-3\gamma}$, this kind of matter is expected to dominate at early times, and the associated solutions give information about the early evolution of the universe. One of the results in [289] is that a necessary condition for $\dot{a} = 0$ when spatial curvature is negligible is $\omega = -6M\varphi/A$, where A and M are positive constants, yielding a negative kinetic term for φ (see Eq. (134)). A thorough qualitative study of the case in which $\omega(\varphi)$ is a monotonic but otherwise arbitrary function of φ was presented in [372], where the existence of nonsingular solutions in theories which agree with GR in the weak field limit was proved.

The first term on the left hand side of Eq. (135) suggests that the gravitational constant is not actually a constant but varies with φ^{-1} . Based on this idea, a generalization of scalar–tensor theories (the so-called hyper-extended scalar–tensor) was advanced in [397]. The Lagrangian associated to these theories is given by

$$\mathcal{L} = -G(\varphi)^{-1}R + \frac{\omega(\varphi)}{\varphi}\varphi_{,\mu}\varphi^{,\mu} + 16\pi\mathcal{L}_{\text{matter}},$$
(147)

which reduces to Eq. (134) when $G(\varphi) = 1/\varphi$. Sufficient conditions on $G(\varphi)$, $\omega(\varphi)$, and their derivatives in order to have bouncing cosmological solutions were given in [159], generalizing the work of [230] for the case of ST theories.

Another descendant of the original ST theory are the multiscalar-tensor (MST) theories [112], which are the generic product of a compactification process of a higher-dimensional theory. The scalar content of a given MST theory depends on details of the internal manifold that results from compactification (usually gauge fields are set to zero in cosmological applications). Typically, one or more fields are associated with the size of the extra dimensions. In string theory, the coupling constants depend on the expectation value of massive scalar fields (called moduli fields) also associated with the size and shape of the extra dimensions, the most popular example of them being the dilaton. The moduli are an inescapable ingredient of string theory, hence several problematic issues raised by them must be confronted, such as stabilization, overcritical density, and violations of the Equivalence Principle. Cosmological solutions of low-energy string theories have been extensively studied (see [257] for a review). Needless to say, the results depend on the field content, which in turn depends on the given string theory under scrutiny.

A possible way to parameterize an action of a MST theory is [111]

$$\mathcal{L} = \sqrt{-g} \left[\phi R - \omega \frac{\phi_{,\rho} \phi^{,\rho}}{\phi} - \phi^n \psi_{,\rho} \psi^{,\rho} - \chi_{,\rho} \chi^{,\rho} \right] + \mathcal{L}_{\text{matter}}.$$
(148)

This Lagrangian represents pure multidimensional theories when $\psi = \text{constant}$, $\chi = \text{constant}$, and $\omega = (1 - d)/d$, where *d* is the number of compactified dimensions (assuming that they have the topology of a torus). The same case but with $\psi \neq$ constant and n = -2/d + 1 corresponds to a two-form gauge field in higher dimensions. If this field is conformal, it is associated with a (d + 4)/2-form, leading to n = -2/d. In the case of string theory, $\omega = -1$, and the field ψ is associated with a three-form field $H_{\mu\nu\lambda}$, leading to n = -1. The scalar χ is related to another three-form field coming from the R-R sector of type IIB superstring theory.

The existence of bouncing solutions for this Lagrangian in a vacuum and in the presence of radiation for the FLRW geometry for all values of the three-curvature and for arbitrary values of ω and n has been studied in [111]. The results show that generically there is a bounce for n < 1 and $\omega < 0$.

3.3.1. Corrections coming from string theory

Superstring theory is a candidate for a unified theory of the fundamental interactions, including gravity [46,368]. Since the fundamental objects in this theory are at least one-dimensional, geodesics of point particles are replaced by worldvolumes. It is a valid question then to ask whether string theory has anything to say about the singularity problem. In this regard, it must be noted that in string theory, gravitational excitations are defined on a fixed metric background. Since singularities in general relativity are boundaries of space-time, which are a consequence of the dynamics governing its structure, a fixed manifold is certainly a restriction. Yet another difficulty is the breakdown of string perturbation theory in the regime of interest [66]. However, we have seen in the previous section that incorporation of the massless degrees of freedom (corresponding to the lowest order EOM), which applies on scales below the string scale and above those where string symmetries are broken, may smooth out the singularity. One could go further and include higher-order corrections in the action of string theory. There are two types of corrections. First, there are the classical corrections arising from the finite size of the strings, when the fields vary over the string length scale, given by $\lambda_s = \sqrt{\alpha'}$. These terms are important in the regime of large curvature, and lead to a series in α' (the inverse of the tension of the string). Then there are the loop (quantum) corrections. The loop expansion is parameterized by powers of the string coupling parameter $e^{\phi} = g_{string}^2$, which is a time-dependent quantity in cosmological models. In the so-called strong coupling regime, the dilaton becomes large and quantum corrections are important.

The effective action at the one-loop level is given by (see for instance [18])

$$S = \int d^4 x \sqrt{-g} \left\{ \frac{R}{2} + \frac{1}{4} (\nabla \phi)^2 + \frac{3}{4} (\nabla \sigma)^2 + \frac{1}{16} [\lambda e^{\phi} - \delta \xi(\sigma)] R_{GB}^2 \right\},$$
(149)

where ϕ is the dilaton, σ is a modulus field, and $\lambda = 2/g^2$ (g is the string coupling), δ is proportional to the 4-d trace anomaly, and $\xi(\sigma) = \ln(2e^{\sigma}\eta^4(ie^{\sigma}))$, where η is the Dedekind function. The correction to the gravitational term is given in terms of the Gauss-Bonnet invariant (see [145,232,306]),

$$R_{CB}^2 = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

The EOM that follow from this action in the case of a FLRW flat spacetime with the metric $g_{\mu\nu} = \text{diag}(1, -e^{2\omega}\delta_{ij})$ are [18]⁵²

$$3\dot{\omega}^2 - \frac{3}{4}\dot{\sigma}^2 - \frac{1}{4}\dot{\phi}^2 + 24\dot{f}\dot{\omega}^3 = 0,$$
(150)

$$2\ddot{\omega} + 3\dot{\omega}^2 + \frac{3}{4}\dot{\sigma}^2 + \frac{1}{4}\dot{\phi}^2 + 16\dot{f}\dot{\omega}^3 + 8\ddot{f}\dot{\omega}^2 + 16\dot{f}\dot{\omega}\ddot{\omega} = 0,$$
(151)

$$\ddot{\sigma} + 3\dot{\omega}\dot{\sigma} + \delta\frac{\partial\xi}{\partial\sigma}\dot{\omega}^2(\dot{\omega}^2 + \ddot{\omega}) = 0, \tag{152}$$

$$\ddot{\phi} + 3\dot{\omega}\dot{\phi} - 3\lambda\epsilon^{\phi}\dot{\omega}^{2}(\dot{\omega}^{2} + \ddot{\omega}) = 0, \tag{153}$$

where $f = \frac{1}{16}(\lambda e^{\phi} - \delta \xi(\sigma))$. These equations are not linearly independent due to the conservation of $T_{\mu\nu}$. It was shown in [18] that there are solutions with bounce for $\delta < 0$, which interpolate between an asymptotically flat and a slowly expanding universe with a period of rapid expansion. The bounce is essentially due to the violation of the strong energy condition by the modulus field (the dilaton playing an unimportant role). In a subsequent paper [352] it was shown that non-singular solutions can be obtained under the assumptions that ξ is a smooth function that has a minimum at some point σ_0 , and grows faster than σ^2 for $\sigma \to \pm \infty$, and $\delta > 0$. However, these solutions were later shown to be generically unstable for tensor perturbations [235]. Less symmetric models (Bianchi I [236] and Bianchi IX [418]) were also studied for this action, confirming the findings of [235].

Another attempt to avoid the singularity is to consider the effect of matter terms to the action of string theory (see [10]). In [401] an action including dilaton, axion and one modulus field was considered along with matter (radiation or a "stringy" gas) and higher-order dilaton corrections in a flat FLRW background in d dimensions. In this case, the results of [401] show that the energy densities of matter, axion and modulus are strongly suppressed in the inflationary phase driven by the dilaton, and hence higher-order corrections coming from this field take the system through a graceful exit.

Yet another model inspired in string theory is the so-called ekpyrotic universe and its extension, the cyclic universe which will be discussed in Section 10.2.4.53

3.3.2. String pre-big bang

A very-well developed example of the string cosmology approach is the so-called "pre-big bang" [179], which we shall call "string pre-big bang" (SPBB), to differentiate it from similar models not coming from string theory (see Section 3.2). There are two properties of string theory that can be expected to play an important role in cosmology [405]. First, in the

⁵² See [144] for the case of nonzero spatial curvature.

⁵³ There are more considerations about singularities and bounces in string theory, to wit AdS/CFT correspondence [117], string gas cosmology [42], and tachyon condensation [376].

157

short-distance regime, a fundamental length λ_s is expected to arise, thus introducing an ultraviolet cut-off and bounding physical quantities such as H^2 and a. Hence a bounce may be expected. Second, as we discussed before, at lower energies, the action of string theory is not Einstein's but a (multi)scalar-tensor theory, where one of the scalar fields is the dilaton, which controls the coupling constants. If these are really constant today (see [403]), the dilaton must be seated at the bottom of its potential, but it may have evolved in cosmological times. The idea of the SPBB is that during cosmological evolution, the kinetic term of the dilaton drove a period of deflation (or inflation, depending on whether we consider the Einstein frame or the string frame) "before the big bang" (that is, in the contracting phase),⁵⁴ which can solve the horizon and flatness problems [172]. In this approach, the universe starts from a perturbative state, passes through a high-curvature and highcoupling stage, and then (hopefully) enters the radiation-dominated FLRW evolution. Duality symmetries present in the low-energy action of string theory are invoked to support this line of reasoning [398]: in the isotropic case, the gravidilaton EOM in the FLRW setting are invariant under a time inversion,

$$t \rightarrow -t \Rightarrow H \rightarrow -H$$

 $\dot{\phi} \rightarrow -\dot{\phi},$

and under the duality transformation

$$a
ightarrow \tilde{a} = a^{-1},$$

 $\phi
ightarrow \tilde{\phi} = \phi - 6 \ln a$

(compare with the Weyl transformation, Eq. (98)). These transformations relate four branches of the solution (PBB, and postbig-bang expansion and contraction). In particular, to any expanding solution with decreasing curvature (such as those in the standard cosmological model), duality associates an accelerated contracting solution (see Fig. 6). It is this pairing (which is possible only in the presence of the dilaton) that supports the whole idea of the SPBB. One of the issues of this idea is the joining of the two phases through the putative singularity (the graceful exit problem). It has been proved in [228] that the graceful exit transition from the initial phase of inflation to the subsequent standard radiation dominated evolution must take place during a "string phase" of high curvature or strong coupling is actually required. The corrections to the lowest-order lagrangian can be parameterized as [85]

$$\mathcal{L}_{c} = \mathcal{L}_{\alpha'} + \mathcal{L}_{q}$$

where

$$\frac{1}{2}\mathcal{L}_{\alpha'} = e^{-\phi} \left(\frac{1}{4} R_{GB}^2 - \frac{1}{2} (\nabla \phi)^4 \right), \tag{154}$$

and \mathcal{L}_q designates quantum loop corrections. Several forms of \mathcal{L}_q were studied in [85]. The existence of a bounce in the Einstein frame, yielding a solution to the graceful exit problem, was shown by numerical integration of the EOM in [85] for the case $\mathcal{L}_q = -2(\nabla \phi)^4$, $\mathcal{L}_q = -2(\nabla \phi)^4 + R^2/3$, and for the two-loop correction $\mathcal{L}_q = 2e^{\phi}R_{GB}^2$, in all cases by choosing the appropriate sign for the correction.

An even more general form of the corrections was studied in [96], where \mathcal{L}_c was given by

$$\mathcal{L}_{c} = -\frac{1}{4} \mathrm{e}^{-\phi} \left(a R_{GB}^{2} + b \phi (\nabla \phi)^{2} + c G^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi + d (\partial_{\mu} \phi)^{4} \right),$$

and 4b + 2c + d = -4a ($G^{\mu\nu}$ is the Einstein tensor). The quantum corrections were included by adding a suitable power of the string coupling, so the total effective Lagrangian is given by

$$\mathcal{L} = R + (\partial_{\mu}\phi)^{2} + \mathcal{L}_{c} + Ae^{\phi}\mathcal{L}_{c} + Be^{2\phi}\mathcal{L}_{c},$$

and the parameters *A* and *B* set the scale for loop corrections. Solutions with graceful exit were found in [96] for a large range of parameters, but it is very hard to obtain the transition in the weak coupling regime, whilst keeping loop corrections small.

A problem that remains to be solved is stabilization of the dilaton to a constant value (otherwise there would be violations to the Equivalence Principle and to the observed "constancy of the coupling constants"). This was achieved in the previously mentioned articles in a number of ways: (1) by introducing by hand a friction term in the equation of motion of the dilaton, and then coupling it to radiation in such a way as to preserve overall conservation, (2) by "turning off" by hand the quantum Lagrangian by means of a step function, and (3) by the manipulation of the sign and size of higher-loop corrections.

3.4. Appendix: Conformal transformation

Consider the map

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x).$$

(155)

⁵⁴ This idea was also suggested in [317].



Fig. 6. The four branches of low-energy string cosmology backgrounds. Taken from [179].

Then, for the contravariant components:

$$\tilde{g}^{\mu\nu}(x) = \Omega^{-2}(x)g^{\mu\nu}(x).$$
(156)

The conformal transformation of the connection is provided by

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \frac{1}{\Omega} \left(\Omega_{,\mu} \,\delta^{\alpha}_{\nu} + \Omega_{,\nu} \,\delta^{\alpha}_{\mu} - \Omega_{,\epsilon} \,g^{\epsilon\alpha} \,g_{\mu\nu} \right), \tag{157}$$

and for the curvature tensor:

$$\tilde{R}^{\alpha\beta}_{\mu\nu} = \Omega^{-2} R^{\alpha\beta}_{\mu\nu} - \frac{1}{4} \delta^{[\alpha}_{[\mu} Q^{\beta]}_{\nu]}, \qquad (158)$$

where

$$Q^{\alpha}{}_{\beta} \equiv 4 \, \varOmega^{-1} \, (\varOmega^{-1})_{,\,\beta;\,\lambda} g^{\alpha\lambda} - 2 \, (\varOmega^{-1})_{,\,\mu} (\varOmega^{-1})_{,\,\nu} \, g^{\mu\nu} \delta^{\alpha}_{\beta}.$$

Contracting Eq. (158) we get

$$\tilde{R}^{\alpha}_{\ \mu} = \Omega^{-2} R^{\alpha}_{\ \mu} - \frac{1}{2} Q^{\alpha}_{\ \mu} - \frac{1}{4} Q \delta^{\alpha}_{\mu}, \tag{159}$$

and contracting again,

$$\tilde{R} = \Omega^{-2} [R + 6 \,\Omega^{-1} \Box \,\Omega]. \tag{160}$$

A direct comparison of this conformal scalar of curvature and the Weyl scalar equation (97) shows that they coincide (up to a multiplicative factor) if we set

$$\Omega = \exp\left(-\frac{1}{2}\,\varphi\right),\,$$

and Eq. (160) takes the form

$$\tilde{R} = \mathrm{e}^{\varphi} \left[R - 3\Box \varphi + \frac{3}{2} \varphi_{,\mu} \varphi^{,\mu} \right],$$

which is exactly the transformed of the Ricci scalar for the WIST:

$$\tilde{R} = e^{\varphi} R^{(W)}.$$

4. Maxwellian and non-Maxwellian vector fields

4.1. Introduction

The model described by the FLRW geometry with Maxwell's electrodynamics as its source displays a cosmological singularity at a finite time in the past [247]. However, this is not an intrinsic property of the combined electromagnetic and gravitational fields. Indeed, modifications of Maxwell electrodynamics (or, generically, massless vector field dynamics) can generate non-singular spatially homogeneous and isotropic (SHI) solutions of classical GR. We shall examine here two modifications that are relevant to the singularity problem:

- The non-minimal coupling of the EM field with gravity, and
- the self-interaction of the EM field.

These modifications will be introduced by means of Lagrangians which depend nonlinearly on the field invariants or on the space-time curvature. In both cases, the singularity theorems (see Section 1) are circumvented by the appearance of a large, but nevertheless finite, negative pressure in an early phase of the SHI geometry.

4.2. Einstein-Maxwell singular universe

The fact that Maxwell electrodynamics minimally coupled to gravity leads to singular models for the universe in the FLRW framework is a direct consequence of the singularity theorems (see Section 1). Essentially, this can be understood from the examination of the energy conservation law and Raychaudhuri equation, as follows. To be consistent with the symmetries of the SHI metric, an averaging procedure must be performed if electromagnetic fields are to be taken as a source for the EE [395]. As a consequence, the components of the electric \mathscr{E}_i and magnetic \mathscr{H}_i fields must satisfy the following relations:

$$\overline{\mathscr{E}}_i = 0, \qquad \overline{\mathscr{H}}_i = 0, \qquad \overline{\mathscr{E}}_i \, \overline{\mathscr{H}}_j = 0, \tag{161}$$

$$\overline{\mathscr{E}}_i \, \overline{\mathscr{E}}_j = -\frac{1}{3} \, \mathscr{E}^2 \, g_{ij}, \tag{162}$$

$$\overline{\mathscr{H}_{i}\mathscr{H}_{j}} = -\frac{1}{3}\mathscr{H}^{2} g_{ij}.$$
(163)

The symmetric energy-momentum tensor associated with Maxwell Lagrangian is given by

$$E_{\mu\nu} = F_{\mu\alpha} F^{\alpha}{}_{\nu} + \frac{1}{4} F g_{\mu\nu}, \tag{164}$$

in which $F \equiv F_{\mu\nu}F^{\mu\nu} = 2(\mathscr{H}^2 - \mathscr{E}^2)$. Using the above average values it follows that the $T_{\mu\nu}$ reduces to a perfect fluid configuration with energy density ρ_{ν} and pressure p_{ν} given by

$$E_{\mu\nu} = (\rho_{\gamma} + p_{\gamma}) v_{\mu} v_{\nu} - p_{\gamma} g_{\mu\nu}, \tag{165}$$

where

$$\rho_{\gamma} = 3p_{\gamma} = \frac{1}{2}(\mathscr{E}^2 + \mathscr{H}^2).$$
(166)

The fact that both the energy density and the pressure in this case are positive definite for all values of *t* implies the singular nature of FLRW universes. In fact, the solution of EE for the above energy–momentum configuration gives for the scale factor the singular form [354]

$$a(t) = \sqrt{a_0^2 t - \epsilon t^2},\tag{167}$$

where a_0 is an arbitrary constant. We conclude that the space-time singularity in the Einstein-Maxwell system is unavoidable.

4.3. Non-minimal interaction

Most of the articles concerning the interaction of Electrodynamics with Gravitation assume the principle of minimal coupling, which is a direct application of the strong form of the Equivalence Principle. In the absence of stringent limits from observation, ideally we should keep an open mind and consider other possibilities. Non-minimal coupling of the EM field with gravity has recently been applied in cosmology, following the trend initiated by scalar field theories interacting conformally with gravitation. These opened the way to the examination of more general theories, such as those in which curvature is directly coupled with the fields.

There are seven possible Lagrangians for the interaction of the EM field with Gravity which can be constructed as linear functionals of the curvature tensor. They are divided in two classes. Class I is given by:

$$\mathcal{L}_1 = R A_{\mu} A^{\mu},$$
$$\mathcal{L}_2 = R_{\mu\nu} A^{\mu} A^{\nu}.$$

These two Lagrangians are gauge dependent but no dimensional constant must be added since they already have the right dimensionality. As shown in [310] the EOM obtained from \mathcal{L}_2 in Einstein's gravity with the addition of a kinetic term for A^{μ} do not admit a FLRW solution. Thus, in the following we shall limit our analysis to \mathcal{L}_1 .

In Class II, there are five Lagrangians:

$$\mathcal{L}_{3} = R F_{\mu\nu} F^{\mu\nu},$$

$$\mathcal{L}_{4} = R F_{\mu\nu} F^{\mu\nu},$$

$$\mathcal{L}_{5} = R_{\mu\nu} F^{\mu}_{\alpha} F^{\alpha\nu},$$

$$\mathcal{L}_{6} = R^{\alpha\beta\mu\nu} F_{\alpha\beta} F^{\nu\mu},$$

$$\mathcal{L}_{7} = \overset{*}{W}_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu},$$
(168)

where $W^{\alpha\beta\mu\nu}$ is the Weyl tensor and the star in the Weyl tensor means

$$\overset{*}{W}_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu}^{*} = W_{\alpha\beta\mu\nu}^{*} = \frac{1}{2} \eta_{\alpha\beta}^{\ \rho\sigma} W_{\rho\sigma\mu\nu}.$$

These Lagrangians are gauge independent but they all need the introduction of a length ℓ_0 in order to have the correct dimensionality.

Another Lagrangian sometimes studied in the literature that is not explicitly contained in this list is

$$\mathcal{L}_8 = R_{\alpha\beta\mu\nu}^* F^{\alpha\beta} F^{\mu\nu}$$

However, \mathcal{L}_8 is not independent of $(\mathcal{L}_1, \ldots, \mathcal{L}_7)$. Indeed, the double dual $R_{\alpha\beta\mu\nu}^*$ satisfies the identity

$$R_{\alpha\beta\mu\nu}^{*} = R_{\alpha\beta\mu\nu} - 2W_{\alpha\beta\mu\nu} - \frac{1}{2}Rg_{\alpha\beta\mu\nu}, \qquad (169)$$

or, equivalently,

$$R_{\alpha\beta\mu\nu}^{*} = -W_{\alpha\beta\mu\nu} + \frac{1}{2} \left(R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu} \right) - \frac{1}{3} Rg_{\alpha\beta\mu\nu}.$$
(170)

Thus,

$$\mathcal{L}_{8} = -\mathcal{L}_{6} - \frac{1}{3} R \left(g_{\alpha\mu} g_{\beta\mu} - g_{\alpha\nu} g_{\beta\mu} \right) F^{\alpha\beta} F^{\mu\nu} + \frac{1}{2} \left(R_{\alpha\mu} g_{\beta\nu} + g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu} \right) F^{\alpha\beta} F^{\mu\nu}$$
Hence, $\mathcal{L}_{8} = -\mathcal{L}_{6} - \frac{2}{2} \mathcal{L}_{3} - 2\mathcal{L}_{5}$.

4.4. An example of a non singular universe

The first example of a nonsingular universe driven by the nonminimal coupling of EM and gravity was presented in [310], using the \mathcal{L}_1 of the previous section:

$$\mathcal{L} = R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \beta R A_{\mu} A^{\mu}.$$
(171)

As mentioned in Section 4.2, in order to obtain a SHI geometry in the realm of General Relativity having a vector field as a source, an average procedure is needed. In the present non-minimal case there is another possibility, which we shall now explore. Since this theory is not gauge-invariant, it is possible to find a non-trivial solution for A_{μ} such that $F^{\mu\nu}$ vanishes.

The equations of motion that follow from the Lagrangian (171) are:

$$(1 + \beta A^2) \left(R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} \right) - \beta \Box A^2 g_{\mu\nu} + \beta (A^2)_{;\mu;\nu} + \beta RA_{\mu}A_{\nu} = -E_{\mu\nu} - T_{\mu\nu},$$
(172)

$$F^{\mu\nu}_{,\nu} = -2\beta RA^{\mu}. \tag{173}$$

From the trace of (172) it follows

 $R = -3\beta \Box A^2,$

which when inserted in the equation of evolution of the electromagnetic field yields a nonlinear equation:

$$F^{\mu\nu}{}_{;\nu} - 6\beta^2 (\Box A^2) A^{\mu} = 0.$$
(174)

The non-linearity induced by the non-minimal coupling with gravity is a generic feature for any field. To obtain a solution in which the geometry is nonsingular for a SHI geometry without imposing an average on the fields [310] we can consider the case in which $F_{\mu\nu}$ is zero. This is possible due to the explicit dependence of the dynamical equations on the vector A_{μ} . We take the vector field A_{μ} of the form

$$A_{\mu} = A(t) \,\delta^0_{\mu}. \tag{175}$$

Defining the quantity Ω by

$$\Omega(t) \equiv 1 + \beta A^2, \tag{176}$$

the set of Eqs. (172) and (173) in a FLRW geometry reduces to the following:

$$3\frac{\ddot{a}}{a} = -\frac{\dot{\Omega}}{\Omega},\tag{177}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\epsilon}{a^2} = -\frac{\dot{a}}{a}\frac{\dot{\Omega}}{\Omega},\tag{178}$$

$$\Box \ \Omega = 0. \tag{179}$$

The last equation implies that $a^3 d\Omega/dt$ is a constant. Thus we set $d\Omega/dt = b a^{-3}$. A particular solution of this set of equations for $\epsilon = -1$ is given by [310]

$$A^{2}(t) = 1 - \frac{t}{a(t)}$$
(180)

$$a(t) = \sqrt{t^2 + \alpha_0^2}$$
(181)

where α_0 is a constant that measures the minimum possible value of the scale factor. When $\alpha_0 = 0$ the system reduces to empty Minkowski space–time in Milne coordinates. For $\alpha_0 \neq 0$ this model represents an eternal universe without singularity and with a bounce.⁵⁵ Notice that in recent years theories with negative energies have been examined in a cosmological context [325]. One way to achieve this goal is by introducing an *ad-hoc* term in the Lagrangian with the wrong sign. In the case of a scalar field this is given as

$$S = \int \sqrt{-g} \left(R - \frac{1}{2} \partial_{\mu} \varphi \, \partial^{\mu} \varphi \right). \tag{182}$$

A fluid with this odd feature can also be obtained by the non-minimal interaction of a vector field with gravity. Indeed, the solution presented in the preceding section can be interpreted as a perfect fluid with negative energy. The equations of motion presented in [310] can be re-written in the form:

$$R_{\mu\nu} = \frac{\Omega_{,\mu;\nu}}{\Omega},\tag{183}$$

were Ω , given by Eq. (176), depends only in time. The structure of the corresponding system of equations is equivalent to the equations of General Relativity in the SHI geometry having as its source the energy–momentum tensor of a perfect fluid with negative energy density and pressure given by

$$p = \frac{1}{3}\rho = -\frac{a_0^2}{a^4}.$$
(184)

In this way, fluids with the "wrong" sign in Einstein's equation can be interpreted as vector fields with non-minimal interaction with gravity.

⁵⁵ This form of the scale factor is similar to Melnikov–Orlov geometry [286], the difference being in the interpretation of the minimum radius *a*₀ and the source of the curvature.



Fig. 7. Plot of the scale factor as a function of *t* for different values of ϵ and α . Taken from [125].

4.5. Nonlinear electrodynamics

As pointed out in the introduction of this Chapter, linear electromagnetism unavoidably leads to a singularity. This situation changes drastically in the case of non-minimal coupling. In this section, we shall deal with another type of theories, in which it is the nonlinearity of the self-interaction of the EM field that provides the necessary conditions for a cosmological bounce to occur. The theories that will be examined are described by Lagrangians which are arbitrary functions of the invariants *F* and *G* that is $\mathcal{L} = \mathcal{L}(F, G)$, where $F = F_{\mu\nu}F^{\mu\nu}$, $G \doteq \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$. Their corresponding energy momentum tensor, computed from Eq. (5) yields

$$T_{\mu\nu} = -4 \,\mathcal{L}_F \, F_\mu^{\ \alpha} F_{\alpha\nu} + (G \mathcal{L}_G - \mathcal{L}) \, g_{\mu\nu}, \tag{185}$$

where $\mathcal{L}_A \equiv d\mathcal{L}/dA$, with A = F, G. It follows that

$$\rho = -\mathcal{L} + G\mathcal{L}_G - 4\mathcal{L}_F \mathscr{E}^2, \tag{186}$$

$$p = \mathcal{L} - G\mathcal{L}_G - \frac{4}{3}(2\mathscr{H}^2 - \mathscr{E}^2)\mathcal{L}_F.$$
(187)

We shall start our analysis by studying a toy model generalization of Maxwell's electrodynamics generated by a Lagrangian quadratic in the field invariants as in [125], that is:

$$L = -\frac{1}{4}F + \alpha F^2 + \beta G^2,$$
(188)

where α and β are dimensionfull constants.⁵⁶

4.5.1. Magnetic universe

In the early universe, matter behaves to a good approximation as a primordial plasma [390,94]. Hence, it is natural to limit our considerations to the case in which only the average of the squared magnetic field \mathcal{H}^2 survives [137,390]. This is formally equivalent to put $\mathcal{E}^2 = 0$ in (162), and physically means to neglect bulk viscosity terms in the electric conductivity of the primordial plasma.

The Lagrangian (188) requires some spatial averages over large scales, such as the one given by Eq. (161)–(163). If one intends to make similar calculations on smaller scales then either more involved Lagrangians should be used, or some additional magnetohydrodynamical effect [393] should be devised in order to achieve correlation [222] at the desired scale. Since the average procedure is independent of the equations of the electromagnetic field we can use the above formulae

⁵⁶ If we consider that the origin of these corrections come from quantum fluctuations then the value of the constants α and β are fixed by the calculations made by Heisenberg and Euler.
(161)–(163) to arrive at a counterpart of expression (165) for the non-Maxwellian case. The average energy–momentum tensor is identical to that of a perfect fluid (165) with modified expressions for the energy density ρ and pressure p, given by

$$\rho = \frac{1}{2} \mathscr{H}^{2} (1 - 8 \alpha \mathscr{H}^{2}),$$

$$p = \frac{1}{6} \mathscr{H}^{2} (1 - 40 \alpha \mathscr{H}^{2}).$$
(189)
(190)

Inserting expressions (189)-(190) in the conservation equation (2) yields

$$\mathscr{H} = \frac{\mathscr{H}_0}{a^2},\tag{191}$$

where \mathcal{H}_0 is a constant. With this result, Eq. (4) leads to

$$\dot{a}^2 = \frac{\mathscr{H}_0^2}{6\,a^2} \left(1 - \frac{8\alpha\,\mathscr{H}_0^2}{a^4} \right) - \epsilon.$$
(192)

Since the right-hand side of Eq. (192) must not be negative it follows that, for $\alpha > 0$ the scale factor a(t) cannot be arbitrarily small regardless of the value of ϵ . The solution of Eq. (192) is implicitly given as

$$t = \pm \int_{a_0}^{a(t)} \frac{\mathrm{d}z}{\sqrt{\frac{\mathscr{H}_0^2}{6z^2} - \frac{8\alpha\mathscr{H}_0^4}{6z^6} - \epsilon}},$$
(193)

where $a(0) = a_0$. The linear case described by Eq. (167) can be regained from Eq. (193) by setting $\alpha = 0$. For the Euclidean section, expression (193) can be solved as⁵⁷

$$a^{2} = \mathscr{H}_{0} \sqrt{\frac{2}{3}} (t^{2} + 12 \alpha).$$
(194)

From Eq. (191), the average strength of the magnetic field \mathscr{H} evolves in time as

$$\mathscr{H}^2 = \frac{3}{2} \frac{1}{t^2 + 12\,\alpha}.$$
(195)

Expression (194) is singular for $\alpha < 0$, as there exist a time $t = \sqrt{-12\alpha}$ for which a(t) is arbitrarily small. Otherwise, for $\alpha > 0$ at t = 0 the radius of the universe attains a minimum value (see Fig. 7) a_0 , given by

$$a_0^2 = \mathscr{H}_0 \sqrt{8\alpha},\tag{196}$$

which depends on \mathscr{H}_0 . The energy density ρ_{γ} given by Eq. (189) reaches its maximum value $\rho_{\text{max}} = 1/64\alpha$ at the instant $t = t_c$, where

$$t_c = \sqrt{12\,\alpha}.\tag{197}$$

For smaller values of t the energy density decreases, vanishing at t = 0, while the pressure becomes negative (see Fig. 8, left panel). Notice that we have a minimum of a(t) along with a minimum of the energy density, entailing a violation of the NEC condition, in accordance with the first row of Table 1.1.

Only for times $t \leq \sqrt{4\alpha}$ the non-linear effects are relevant for the normalized scale-factor, as shown in Fig. 8, left panel. Indeed, the solution (194) yields the standard expression (167) of the Maxwell case at the limit of large times. Notice that the energy-momentum tensor (185) is not trace-free for $\alpha \neq 0$. Thus, the equation of state $p_{\gamma} = p_{\gamma}(\rho_{\gamma})$ is no longer that of Maxwell's; it has instead a term proportional to the constant α , that is

$$p = \frac{1}{3}\rho - \frac{16}{3}\alpha \mathscr{H}^4.$$
 (198)

This scenario has been generalized in several ways in [93]. First, the general expression for the scale factor was shown to be

$$a(t) = a_0 (4\alpha_0^2 t^2 + 4\alpha_0 \beta_0 t + 1)^{1/4},$$
(199)

where

$$\alpha_0 = \sqrt{\frac{2}{3}} \mathscr{H}_0, \qquad \beta_0 = \pm \sqrt{1 - 8\alpha \mathscr{H}_0}.$$

Eq. (194) follows as a particular case from Eq. (199), which describes a bounce with

$$a_{\min} = a_0 (8\omega \mathscr{H}_0^2)^{1/4}, \qquad t_{\min} = -\beta_0/(2\alpha_0), \qquad \mathscr{H}_{\min} = \frac{1}{2\sqrt{2\alpha}}, \qquad \rho_{\min} = 0.$$

Solutions of this model with the addition of a cosmological constant Λ were also discussed in [93]. It was shown that nonsingular solutions are possible both for a constant Λ , and for certain choices of $\Lambda = \Lambda(t)$.

⁵⁷ Nonsingular solutions in Bianchi universes with nonlinear electrodynamics as a source were studied in [170].



Fig. 8. Left panel: time dependence of ρ and p for $\epsilon = 0$ and $\alpha > 0$. Right panel: scale factor for $\epsilon = 0$ and $\alpha > 0$ (full line), and for Mawxell's case (dashed line). Taken from [125].

4.5.2. Born-Infeld electrodynamics

A widely studied EM theory is that proposed by Born and Infeld, with Lagrangian

$$L_{BI} = \beta^2 \left(1 - \sqrt{X} \right) \tag{200}$$

where

$$X \equiv 1 + \frac{1}{2\beta^2}F - \frac{1}{16\beta^4}G^2.$$
 (201)

Note that, following Born–Infeld's original work, a constant term has been added in the Lagrangian in order to eliminate a cosmological constant and to set the value of the Coulomb-like field to be zero at the infinity. Using Eq. (186) for the energy density we obtain

$$\rho = \frac{\beta^2}{\sqrt{X}} \left(1 - \sqrt{X} + \frac{\mathscr{H}^2}{\beta^2} \right) \tag{202}$$

and for the pressure

$$p = \frac{\beta^2}{\sqrt{X}} \left(\sqrt{X} - \beta^2 + \frac{2}{3} \frac{\mathscr{E}^2}{\beta^2} - \frac{1}{3} \frac{\mathscr{H}^2}{\beta^2} \right).$$
(203)

A straightforward calculation of ρ + 3*p* shows that this theory cannot yield a nonsingular universe.

4.5.3. Bouncing in the magnetic universe

The "magnetic universe" displays a very interesting property due to nonlinear dynamics: its energy density can be interpreted as composed of *k* non-interacting fluids, in the case in which the dynamics is provided by the polynomial

$$\mathcal{L} = \sum_{k} c_k F^k, \tag{204}$$

where $k \in \mathbb{Z}$. The conservation of the energy–momentum tensor projected in the direction of the co-moving velocity $v^{\mu} = \delta_0^{\mu}$ yields

$$\dot{\rho} + (\rho + p)\theta = 0. \tag{205}$$

From the expression for the energy density and pressure given in Eqs. (186) and (187) with $\mathscr{E} = 0$ we get that $\rho = \sum_k \rho_k$, and $p = \sum_k p_k$ where

$$\rho_{k} = -c_{k} 2^{k} \mathscr{H}^{2k}$$

$$p_{k} = c_{k} 2^{k} \mathscr{H}^{2k} \left(1 - \frac{4k}{3}\right),$$
(206)

in such a way that we can associate to each power of k an independent fluid characterized by ρ_k and p_k , with an EOS

$$p_k = \left(\frac{4k}{3} - 1\right)\rho_k.$$

Inserting the total energy density and pressure (from the sum of ρ_k and p_k in Eqs. (206)) in the conservation equation (205) we obtain

$$\mathcal{L}_F\left[(\mathscr{H}^2)^{\cdot} + 4\mathscr{H}^2\frac{\dot{a}}{a}\right] = 0.$$
(207)

The important result that this equation shows is that each k-fluid is separately conserved, since the dependence of the conservation equation on the specific form of the Lagrangian factors out, in such a way that \mathscr{H} evolves with the scale factor as

$$\mathscr{H} = \frac{\mathscr{H}_0}{a^2} \tag{208}$$

for any \mathcal{L} of the form given in Eq. (204).

4.5.4. Two-fluid description

It follows from Eqs. (189)–(191) that in the case of the nonlinear Lagrangian given by Eq. (188) it is not possible to write an equation of state relating pressure to energy density. This is a drawback if we want to use a fluid description of the averaged electromagnetic field. In order to circumvent such difficulty a two-fluid description can be adopted, because of the remarkable fact that there exists a separate law of conservation for each component of the fluid, as we saw above. The fact that the dynamical equation for \mathcal{H} factors (see Eq. (207)) means that the fluids are conserved independently: the energy–momentum tensor can be separated into two pieces, each representing a perfect fluid which is conserved independently. In other words, there is no interaction between fluids 1 and 2. We shall see in Section 11.2 that the analysis of the stability of the non-singular universe described in this section is more transparent when using the two-fluid description. This case can be generalized to a multi-component fluid, but we shall restrict here to the 2-fluid application for a pure magnetic field.

In order to get a better understanding of the properties of the cosmic geometry controlled by the magnetic field let us analyze the case in which the spatial section is closed ($\epsilon = 1$). The crucial equations for such analysis are the conservation law, the Raychaudhuri equation for the expansion and the Friedman equation, that is:

$$\dot{\rho} + (\rho + p)\,\theta = \mathbf{0},\tag{209}$$

$$\dot{\theta} + \frac{1}{3}\theta^2 = -\frac{1}{2}(\rho + 3p).$$
 (210)

$$\rho = \frac{1}{3}\theta^2 + \frac{3}{a^2}.$$
(211)

In the magnetic universe we have

$$\rho = \frac{\mathscr{H}_0^2}{2a^4} \left(1 - 8\alpha \,\frac{\mathscr{H}_0^2}{a^4} \right). \tag{212}$$

A necessary condition for the existence of a bounce is given by the vanishing of the expansion factor for a given value of *t*. This leads to an algebraic equation of third order in $x \equiv a_b^2$:

$$x^{3} - \frac{\mathscr{H}_{0}^{2}}{6}x^{2} + \frac{4}{3}\alpha\mathscr{H}_{0}^{4} = 0.$$
(213)

Using the fact that α is a very small parameter, it can be shown that this equation has three real solutions. Two of them are positive and the third is negative. Thus we retain only the positive solutions which will be called X_1 and X_2 . The important quantity for our analysis is contained in the expression

$$\rho_b + 3p_b = \frac{\mathscr{H}_0^2}{x^4} \left(x^2 - 24\alpha \mathscr{H}_0^2 \right). \tag{214}$$

Thus, at one of the points, say X_1 there is a local maximum for the scale factor; and at the other, X_2 there is a minimum for $x^2 < 24\alpha \mathscr{H}_0^2$. Note that at the bounce (where $\theta = 0$), there is an extremum of the total energy: $\dot{\rho}_b = 0$. The analysis of the second derivative in the bounce depends on the location of X_2 through the equations:

$$\ddot{\rho}_{b} = \frac{1}{3} \frac{\mathscr{H}_{0}^{4}}{x^{8}} \left(x^{2} - 16\alpha \mathscr{H}_{0}^{2} \right) \left(x^{2} - 24\alpha \mathscr{H}_{0}^{2} \right).$$
(215)

At $x = X_1$ the density is a minimum. For $x = X_2$ the extremum depends on the location of the bounce with respect to the point in which the quantity $\rho + p$ changes sign. For the case in which $16\alpha \mathcal{H}_0^2 < X_2 < 24\alpha \mathcal{H}_0^2$, it follows that the density

has a maximum at X_1 . On the other hand if $X_2 < 16\alpha \mathscr{H}_0^2$ it is a minimum. To understand completely the behavior of the energy density the existence of other critical points for ρ must be addressed. This is controlled by Eq. (209). Thus, the extra extremum (which are not bounce or turning points) occur at *x* such that

$$\rho + p = \frac{2}{3} \frac{\mathscr{H}_0^2}{x^4} \left(x^2 - 16\alpha \mathscr{H}_0^2 \right) = 0, \tag{216}$$

that is, at points in which the scale factor takes the value $\sqrt{16\alpha \mathscr{H}_0^2}$. Direct inspection shows that these are points of maximum density.

Another consequence of nonlinear electromagnetism in cosmology is the occurrence of a cyclic universe, as will be discussed in Section 10.2.2.

4.6. Appendix

4.6.1. Repulsive gravity

A peculiar result which may provide a framework to generate cosmological scenarios without singularity comes from the nonminimal interaction of EM with gravity, rendering gravity repulsive. The theory is defined by

$$L = \sqrt{-g} \left\{ R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta R A_{\mu} A^{\mu} \right\},$$
(217)

where β is a dimensionless constant. This Lagrangian is not gauge-invariant and can be interpreted in terms of a photon having a mass (and also an additional polarization state) which depends on the curvature of the geometry.

Variation of $g_{\mu\nu}$ and A_{μ} yield the equations of motion:

$$\left(\frac{1}{\kappa} + \beta A^2\right) G_{\mu\nu} = \beta g_{\mu\nu} \Box A^2 - \beta A^2_{,\mu;\nu} - \beta R A_{\mu} A_{\nu} - E_{\mu\nu},$$
(218)

$$F^{\mu\nu}_{;\nu} = -2\beta RA^{\mu}, \tag{219}$$

where $E_{\mu\nu}$ is Maxwell's energy–momentum tensor given by Eq. (164). As will be shown next, this set of equations allows a renormalization of the gravitational constant. Consider for instance the case in which $A_{\mu}A^{\mu} = Z = \text{constant} \neq 0$. Then

$$\left(\frac{1}{\kappa} + \beta Z\right) G_{\mu\nu} = -\beta R A_{\mu} A_{\nu} - E_{\mu\nu}.$$
(220)

Taking the trace of this equation we obtain R = 0, and inserting this result back into Eq. (220) we get

$$R_{\mu\nu} = -\tilde{\kappa}E_{\mu\nu}$$

where the renormalized constant $\tilde{\kappa}$ is given by

$$\frac{1}{\tilde{\kappa}} = \frac{1}{\kappa} + \beta Z.$$

Thus, Eqs. (218) and (219) can be written as

$$R_{\mu\nu} = -\tilde{\kappa}E_{\mu\nu}, \qquad F_{;\nu}^{\mu\nu} = 0, \tag{221}$$

which are nothing but Maxwell's electrodynamics minimally coupled to gravity with a re-normalized gravitational coupling plus the condition $A_{\mu}A^{\mu}$ = constant = Z.

The addition of other forms of neutral matter, such that the corresponding energy-momentum tensor is traceless, takes the Lagrangian to

$$L = \sqrt{-g} \left\{ \frac{1}{\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta R A_{\mu} A^{\mu} + L^{(m)} \right\},$$
(222)

where $L^{(m)}$ represents the Lagrangian for all other kinds of matter such that $T^{(m)}_{\mu\nu}g^{\mu\nu} \equiv T^{(m)} = 0$. The equations of motion in this case are given by

$$\left(\frac{1}{\kappa} + \beta A^{2}\right)G_{\mu\nu} = \beta \Box A^{2}g_{\mu\nu} - \beta A^{2}_{,\mu;\nu} - \beta RA_{\mu}A_{\nu} - E_{\mu\nu} - T^{m}_{\mu\nu}$$

$$F^{\mu\nu}_{,\nu} = -2\beta RA^{\mu}.$$
(223)
(224)

Taking again the case $A_{\mu}A^{\mu}$ = constant, yields R = 0. Then Eqs. (223)–(224) take the reduced form

$$R_{\mu\nu} = -\tilde{\kappa}E_{\mu\nu} - \tilde{\kappa}T^{(m)}_{\mu\nu}$$
$$F^{\mu\nu}_{;\nu} = 0,$$

where $\tilde{\kappa}$ was given above. Thus, the renormalization of the gravitational constant by non-minimal coupling represented by the presence of the term $RA_{\mu}A^{\mu}$ in the Lagrangian in the state where $A^{\mu}A_{\mu}$ is constant is still valid in the presence of matter with null trace.⁵⁸

⁵⁸ Note that this model provides a mechanism for a bounce, but needs to be modified to account for the correct large-scale behavior.

4.6.2. Global dual invariance

While observation must be the ultimate judge of the choice among possible couplings, if it scarce or not available, we can resort to criteria coming from theoretical considerations. One of them is related to the invariance of the Lagrangian under a given transformation, such as the dual rotation. A dual map is a transformation on the set of the bi-tensors $F_{\mu\nu}$ such that

$$F_{\mu\nu} \to F'_{\mu\nu} = \cos\theta F_{\mu\nu} + \sin\theta F^*_{\mu\nu}.$$
(225)

Classical Maxwell's electrodynamics is invariant under such transformation only if the angle θ is constant. In a Minkowskian background it is not possible to implement such invariance for a local map $\theta = \theta(x)$. However, this can be achieved in the case of non-minimal coupling of the electromagnetic field with the metric of a non-flat geometry. In fact, using the identities

$$F_{\mu\alpha} F^{\alpha\nu} - F^*_{\mu\alpha} F^{*\,\alpha\nu} = -\frac{F}{2} \,\delta^{\nu}_{\mu}$$
$$F_{\mu\alpha} F^{*\,\alpha\nu} = -\frac{G}{4} \,\delta^{\nu}_{\mu}$$

it can be shown that the combined Lagrangian:

$$L_{\rm DI} = L_5 - \frac{1}{4} L_3 = \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) F^{\mu}_{\alpha} F^{\alpha\nu}$$
(226)

is invariant under local dual rotations: $\tilde{L}_{Dl} = L_{Dl}$. This is a remarkable property which has no counterpart in the flat space limit.

5. Viscosity

A full knowledge of the global properties of the universe cannot be achieved without giving a description of the thermodynamics of the cosmic fluid. In the last decades, this task was addressed in three distinct periods. In the first period the universe was treated as a system in equilibrium in which all global processes were described by classical reversible thermodynamics, in such a way that total entropy was conserved. The salient feature of this phase was the development of the standard cosmological model, which comprises homogeneous and isotropic FLRW geometry, and the characterization of the matter content of the universe as a one-component perfect fluid in equilibrium. In order to solve the EE, the energy density ρ and the pressure p were considered functions of cosmological time only, and they were related by a linear EOS $p = \lambda \rho$. The FLRW models generated in this way share the common property of having an initial singularity (with $\lambda > -1/3$).

Later, it was realized [135] that the validity of thermal equilibrium near the initial singularity is perhaps too strong an assumption. A second phase then started, in which the description of the cosmic fluid was improved by allowing viscous processes. Some of the motivations for this alteration are the following:

- The examination of the possible role of viscosity in the dissipation of eventual primordial anisotropies (chaotic cosmologies),
- The effect on the existence and/or the form of the singularity,
- The application in cosmology of results obtained from non-equilibrium thermodynamics.

In 1973 a FLRW cosmological model without singularity was presented [299] (see also [246]), using a viscous fluid as a source. The energy-momentum tensor was given by

$$T_{\mu\nu} = (\rho + p) v_{\mu} v_{\nu} - p g_{\mu\nu},$$

in which $p = p_{th} - \zeta \theta$; where p_{th} is the thermodynamical pressure, ζ is a viscous coefficient and θ is the three times Hubble parameter, which is exactly the case of the energy–momentum tensor representing particle creation [51]. The SEC in this case is given by the inequalities

$$\rho + p_{th} > 0$$

and

$$\rho + 3p_{th} > 0,$$

which are weaker than the correspondent ones in the case of a perfect fluid, hence allowing for the absence of the singularity. The solution found in [299] is nonsingular, and past-eternal.

More general forms for the dependence of viscous quantities have been investigated for arbitrary Stokesian regimes in which fluid parameters become more general (for instance nonlinear) functions of the expansion. With these modifications, there are non-singular cosmological solutions, but they may suffer from a possibly worse disease than the initial singularity: they are unstable and display non-causal propagation. In fact, the instability of the model in [299] under homogeneous perturbations was proven by the analysis made in [51]. It was also proved in [51] that avoidance of the singularity is not generic. In other words, the singularity is not avoided for any type of viscosity (that is, for any dependence of the coefficients of viscosity on the expansion factor).

In this second phase, local equilibrium [345] is still imposed, in such a way that thermodynamical variables are described as if the dissipative fluxes – e.g. heat flux – do not influence local variables like for instance entropy, although as a whole the system is not in equilibrium. As another example, a fluid in the regime

$$\tilde{p} = p + \alpha \theta + \beta \theta^2$$

was analyzed in [311,312], both for $\alpha = \beta = \text{constant}$, and $\alpha = 0$, $\beta = M\rho^m$, with *M* and *m* constants. In the second case, nonsingular solutions were found using tools from dynamical systems analysis.

Let us remark that in general, the imposition of local equilibrium leads to causal difficulties, allowing dissipative signals to travel with infinite velocity of propagation. These causal problems were the focus of the third phase, where extended irreversible thermodynamics was used [220,335]. In this theory, the basic quantities become dependent not only on local variables of classical thermodynamics but also on dissipative fluxes. This has very important consequences, the most important one being the preservation of causal connections for the whole system. In [127], a FLRW universe was studied in this context, the net consequence of the assumption of extended irreversible thermodynamics being to provide an additional equation of motion for the non-equilibrium pressure π , with $p = p_{\text{th}} + \pi$, given by

$$\tau_0 \dot{\pi} + \pi = -\xi \theta \tag{227}$$

(where τ_0 is the relaxation time) which preserves the causal structure. Thus, contrary to the previous case in which the viscous term is assumed to be a polynomial in θ , here it must obey Eq. (227). The other quantities relevant to thermodynamics (that is, the entropy flux s^{α} and the particle flux per unit of proper volume n) are determined by

$$n\dot{s} = rac{\pi^2}{\xi T}, \qquad heta = -rac{\dot{n}}{n}.$$

Assuming an EOS given by $p_{th} = \lambda \rho$, the cases $\xi = \text{constant}$, and $\xi = \beta \rho$, (with $\beta = \text{constant}$) were analyzed in [127], always with $\tau_0 = \text{constant}$, and nonsingular solutions were discovered in both cases, for $\lambda = 0$ and $\lambda = 1/3$. The relevant equations of this system can be put in the form of an autonomous planar system:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -3/2(1+\lambda)\theta^2 - \frac{\pi}{2} + \frac{(1+\lambda)}{2}\Lambda,$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} = -\frac{1}{\tau_0}(1+3\zeta\theta),$$
(228)

where Λ is the cosmological constant. The set of integral curves of this system was studied in [316], where it was shown that the solution found in [127] is stable.

Bifurcations in the early cosmos

Quadratic dissipative processes were analyzed from a new perspective in [313], where it was shown that dissipative processes may lead to the appearance of bifurcations. This is a consequence of the application of a theorem due to Bendixson [17] to the system of EE that describes a universe with curvature controlled by a dissipative fluid. Indeed, let us consider a planar autonomous system that contains a parameter, say σ , of the form

$$\dot{x} = F(x, y; \sigma)$$

$$\dot{y} = G(x, y; \sigma),$$
(229)

where the functions *F* and *G* are non-linear and the parameter σ has a domain \mathcal{D} . Applying methods of qualitative analysis to this system and restricting to the two-dimensional plane Γ of all integrals of this system, one arrives to the notion of "elliptical" and "hyperbolic" sectors, that characterize, as the names indicates, the behavior of the integral curves in the neighborhood of a multiple equilibrium point (that is, an isolated points that is a zero of both *F* and *G*). Let us call \mathcal{E} and \mathcal{H} the number of elliptical and hyperbolic sectors of a given equilibrium point $M \equiv (x_0, y_0)$ of Γ , respectively. Then the Poincaré index is defined by the formula

$$I_P = \frac{\mathcal{E} - \mathcal{H}}{2} + 1.$$

This is a measure of the topological properties of the integral curves in the phase plane Γ . If above a certain value σ_c of \mathcal{D} the topological properties of the system (229) change, then there is an abrupt change of behavior of the physical system in the vicinity of the unstable equilibrium point. The crucial consequence of the above-given theorem is the appearance of indeterministic features. In [313] this theorem was applied to spatially homogeneous and isotropic cosmological models, whose dynamics is described by a planar autonomous system, given by

$$\dot{\rho} = -\gamma \rho \theta + \alpha \theta^2 + \beta \theta^3,$$

$$\dot{\theta} = -\frac{3\gamma - 2}{2} \rho + \frac{3\alpha}{2} \theta + \left(\frac{3\beta}{2} - \frac{1}{3}\right) \theta^2,$$
(230)

where σ (referred to in the theorem) can be either α , β or γ , and the energy–momentum tensor is

$$T_{\mu\nu} = (\rho + \tilde{p}) v_{\mu} v_{\nu} - \tilde{p} g_{\mu\nu},$$

where

$$\tilde{p} = p_{th} + \alpha \,\theta + \beta \,\theta^2,$$

with $p_{\text{th}} = (\gamma - 1) \rho$.

The viscous terms (parameterized by α and β) can be a phenomenological description of particle creation in a nonstationary gravitational field as proposed in [406,425]. Applying the methods of qualitative analysis to the system given in Eq. (230) it was shown in [313] that for $\gamma - 3\beta < 0$, the Poincaré index $I_P(\mathcal{B}) = -1$ (saddle point); for $\gamma - 3\beta \ge 0$, $I_P(\mathcal{B}) = 1$ (two-tangent node). This situation characterizes a bifurcation in the singular point, when $\rho = \theta = \infty$. This bifurcation, caused by dissipative processes involving quadratic viscous terms generates a high degree of indeterminacy in the development of the solution of EE, which enshrouds the past of this model of the universe. In this case, nothing can be stated about the existence of the initial cosmological singularity.

6. Bounces in the braneworld

Theoretical developments coming from string theory have revived the idea that our universe may have more than 4 dimensions (first considerated by Kaluza in the context of unification of gravity and electromagnetism). Among the multidimensional models, those with one or more branes that live in a bulk space have been thoroughly studied recently (see for instance [273]). In these models, the matter fields are typically confined to a 3-brane in 1 + 3 + d dimensions, while the gravitational field can propagate also in the *d* extra dimensions, which need not be small, or even finite, as shown in one of the models introduced by Randall and Sundrum [347], where for d = 1, gravity can be localized on a single 3-brane even when the fifth dimension is infinite. The Friedmann equation on the brane is modified by high-energy matter terms and also by a term which incorporates nonlocal effects of the bulk onto the brane [54,273]:

$$H^{2} = \frac{\Lambda}{3} + \frac{\kappa^{2}}{3}\rho - \frac{\epsilon}{a^{2}} + \frac{\overline{\kappa}^{4}}{36}\rho^{2} + \frac{1}{3}\left(\frac{\overline{\kappa}}{\kappa}\right)^{4}\mathcal{U}_{0}\left(\frac{a_{0}}{a}\right)^{4},$$
(231)

where ϵ is the 3-curvature, $H = \dot{a}/a$, ρ is the energy density of the matter on the brane, $\bar{\kappa}^2 = 8\pi/\overline{M}_{pl}^3$, \overline{M}_{pl}^3 is the fundamental 5-dimensional Planck mass, $\kappa^2 = 8\pi/M_{pl}^2$, and

$$\Lambda = \frac{4\pi}{\overline{M}_{pl}^3} \left[\overline{\Lambda} + \left(\frac{4\pi}{3\overline{M}_{pl}^3} \right) \lambda^2 \right],$$

where λ is the tension of the brane, and $\overline{\Lambda}$ is the 5-dimensional cosmological constant. Finally, \mathcal{U}_0 is the constant corresponding to the non-local energy conservation equation. This term comes from the projection of the Weyl tensor of the bulk on the brane [273]. From Eq. (231) we see that a necessary condition to have a bounce with $\rho > 0$ in the $\epsilon = 0, -1$ cases is that either $\Lambda < 0$ or $\mathcal{U} < 0$, or both.

The case that includes matter in the bulk, without cosmological constant for a flat FLRW d + 1-dimensional was studied in [165]. A necessary condition in order to have a bounce is that dH/dt > 0, with

$$\frac{dH}{dt} = \frac{\kappa^2}{d}(R+P) - \left(\frac{8\pi G_N}{d-1} + \frac{\kappa^4}{4d}\rho\right)(\rho+p) - \frac{d+1}{d(d-1)}E_0^0,$$
(232)

where (in a notation slightly different from that used in Eq. (231)) κ is the bulk gravitational coupling, G_N the effective Newton constant on the (d + 1)-dimensional brane, E is the projection of the bulk brane Weyl tensor on the brane, and $T^{\mu}_{\nu} = (-R, \vec{P})$ is the projection of the bulk energy–momentum tensor on the brane. It follows from this equation that a necessary condition to have a bounce without resorting to exotic forms of matter (that is, matter that violates $\rho > 0$ or $\rho_p > 0$) is a negative E_0^0 [165]. This is precisely the approach taken in [233,295], where a brane evolving in a charged AdS black hole background was studied. Bouncing solutions were found for both critical ($\Lambda = 0$) and non-critical ($\Lambda \neq 0$) branes, the bounce generically depending on the parameters of the black hole, and on the matter content of the brane.⁵⁹

The abovementioned necessary condition was explicitly checked in the case of the dilaton-gravity braneworld [165], and bouncing solutions were obtained for a a flat FLRW brane in a static spherically symmetric bulk.⁶⁰ This solution describes (in the string frame) a pre-big bang model where the transition between the branches is realized at low curvature and weak coupling, thus providing an example of successful graceful exit without resorting to quantum or "stringy" corrections.

Notice that the extra dimension(s) could be spacelike or timelike. The latter case was analyzed in [374]. The usual incantations [273] for the case of an extra timelike dimension and an homogeneous and isotropic brane lead to [374]

$$H^{2} + \frac{\epsilon}{a^{2}} = \frac{\Lambda}{3} + \frac{8\pi G\rho}{3} - \frac{\rho^{2}}{\bar{M}_{\rm Pl}^{6}} + \frac{C}{a^{4}},$$
(233)

⁵⁹ The bounce in the model presented in [295] was analyzed from the point of view of the causal entropy bound in [285], and its stability was put in doubt in [212].

⁶⁰ Bouncing solutions for a domain wall in the presence of a Liouville potential were found in [103].

where *G* and *A* are the effective gravitational and cosmological constant, respectively, and *M* is the 5-dimensional Planck mass. Notice that the minus sign in front of ρ^2 may lead to a bounce instead of a singularity, since this term grows faster than the others, leading to H = 0, this feature being independent of the equation of state and also of the spatial curvature of the universe. The simplest of these bouncing universes, described by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\rho^2}{\bar{M}_{\rm Pl}^6}$$
(234)

will be discussed in Section 10.2.1, since it may lead to a cyclic universe.

The case with an extra timelike dimension in this scenario was also extended to Bianchi I universes [374], which exhibit an anisotropic bounce as long as the shear scalar $\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ does not grow faster than a^{-8} as *a* goes to zero at the end of the contraction phase. All these results were obtained by neglecting the induced curvature on the brane, which can trigger the formation of a singularity at the beginning or at the end of the evolution [374].

Another model along these lines was introduced in [14], where a "test brane" (i.e. one that does not modify the ambient geometry) moves in a higher-dimensional gravitational background. Using the thin-shell formalism, in which the field equations are re-written as junction conditions relating the discontinuity in the brane extrinsic curvature to its vacuum energy, the motion of domain walls in de Sitter and anti-de Sitter (AdS) time-dependent bulks was discussed. This motion induces a dynamical law for the brane scale factor, and it was shown in [14] that in the case of a clean brane, the scale factor may describe a non-singular universe. In order to build the class of geometries of interest, two copies of (d + 1)-dimensional dS (AdS) spaces *M*1 and *M*2 undergoing expansion were considered. From each of them, one identical *d*-dimensional region Ω_i (i = 1, 2) was removed yielding two geodesically incomplete manifolds with boundaries given by the hypersurfaces $\partial \Omega_1$ and $\partial \Omega_2$. Finally, the boundaries were identified up to an homeomorphism $h : \partial \Omega_1 \rightarrow \partial \Omega_2$. Hence, the resulting manifold that is defined by the connected sum *M*1#*M*2 is geodesically complete. The starting point is the action

$$S = \frac{\ell_{\text{Pl}}^{(3-d)}}{16\pi} \int_{M} d^{d+1}x \sqrt{g} (R - 2\Lambda) + \frac{\ell_{\text{Pl}}^{(3-d)}}{8\pi} \int_{\partial\Omega} d^{d}x \sqrt{\gamma} \mathcal{K} + \sigma \int_{\partial\Omega} d^{d}x \sqrt{\gamma}$$

where the first term is the usual Einstein–Hilbert action with a cosmological constant Λ , the second term is the Gibbons–Hawking boundary term, \mathcal{K}_{MN} is the extrinsic curvature, and σ is the intrinsic tension of the *d*-dimensional brane. The spatial coordinates on $\partial \Omega$ can be taken to be the angular variables ϕ_i , which for a spherically symmetric configuration are always well defined up to an overall rotation. Generically, the line element of each patch can be written as

$$ds^{2} = -dt^{2} + A^{2}(t)[r^{2}d\Omega_{(d-1)}^{2} + (1 - kr^{2})^{-1}dr^{2}]$$

where ϵ takes the values 1 (-1) for dS (AdS), $\Omega^2_{(d-1)}$ is the corresponding metric on the unit d - 1-dimensional sphere, and t is the proper time of a clock measured in higher-dimensional spacetime. In order to analyze the dynamics of the system, the brane is allowed to move radially. Let the position of the brane be described by $x_{\mu}(\tau, \phi_i) \equiv (t(\tau), a(\tau), \phi_i)$, with τ the proper time (as measured by co-moving observers on the brane) that parameterizes the motion, and the velocity of a piece of stress-energy at the brane satisfying $u^M u_M = -1$. With these assumptions the brane will have an effective scale factor $\mathcal{A}^2(t) = a^2(t)\mathcal{A}^2(t)$. The constraint

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \pm \sqrt{1 - \frac{(A\dot{a})^2}{1 - \epsilon a^2}}$$

along with the result of the integration of EE across the boundary (done with the junction conditions) [14] yields two differential equations for *A* and *a*. For the case of a background composed by two patches of dS undergoing expansion, $A(t) = \ell \cosh(t/\ell)$, and $\epsilon = 1$, where $\ell^2 = d(d-1)/|\Lambda|$ is the dS radius. In this case the EOM for the brane is

$$\frac{4\pi}{\binom{d^{(3-d)}(d-1)}{d}} \sigma = \frac{\pm \dot{a}\sinh(t/\ell) + [a\ell\cosh(t/\ell)]^{-1}(1-a^2)}{(1-a^2 - [\ell\dot{a}\cosh(t/\ell)]^2)^{1/2}}$$

Nonsingular analytical solutions of this equation for $\sigma = 0$ can be obtained, while for $\sigma \neq 0$, numerical methods must be used. This latter case also yields bouncing solutions (see Fig. 9).⁶¹

The motion of a test brane in a background produced by a collection of branes was discussed in [237] (so-called mirage cosmology). Adopting spherically-symmetric backgrounds, it was shown that although there is a singularity in the evolution of the 4-d brane, the higher-dimensional geometry is regular. The origin of the singularity on the brane is actually embedding of the brane in the bulk, in such a way that the singularity is smoothed out when the solution is lifted to higher dimensions.

The effect of inflation on a bouncing brane was used in [20] to set limits on the parameters of the braneworld. Specifically, the model consists of a closed FLRW metric embedded in a 5-d conformally flat bulk with one extra timelike dimension, containing a conformally coupled scalar field (the inflaton field) and a radiation fluid, evolving on the brane with corrections due to the bulk. The non-singular bouncing solutions considered were oscillatory and bounded, or initially bounded. They

⁶¹ Another model along this lines can be found in [373].



Fig. 9. Effective scale factor $\mathcal{A}(t)$ plotted for $\bar{\sigma} = 4\pi\sigma/(L_p^{(3-d)}(d-1)) = 10^{-4}$. The insert displays the dependence of \mathcal{A} with $\bar{\sigma}$. Taken from [14].

are in principle stable and would never enter an inflationary phase with an exponential growth of the scale factor since they correspond to periodic orbits of the integrable dynamics in the gravitational sector. The introduction of a massive scalar field, even in the form of small fluctuations, turns non-integrable the dynamics of the system [20]. As a consequence, non-linear resonance phenomena are present in phase space dynamics for certain domains of the parameter space of the models, and the associated dynamical configurations become metastable, allowing the orbits escape to the de Sitter infinity in a finite time. From the conditions for these orbits to happen, limits on the parameters (σ , m, E_0) are set, where σ is the brane tension, m is the mass of the scalar field, and E_0 is a constant proportional to the total energy of the fluid.

Yet another turn in the mirage model was introduced in [180], where the brane moves in an open orbit around a nontrivial spherically-symmetric background. In this model, the brane is moving on a Calabi–Yau manifold generated by a heap of D3-branes, and the mirage effects dominate the evolution of the Universe only at early time, i.e. when the brane moves in the throat of the background manifold. The new feature is the influence of the angular momentum of the test brane on its motion in the higher-dimensional space. In fact, the effective 4-d metric has two parameters: the energy *U* and the angular momentum *L* of the 4-d brane, which determine the form of the orbit. In particular, to have an open orbit in an asymptotically Minkowskian background,

$$L^4 - 4(U+2)U^3 \ge 0.$$

As discussed in [180], the effective metric corresponding to orbits satisfying this constraint display cosmological contraction during the ingoing part of the orbit, expansion during the outgoing part, and a bounce at the turning point.⁶²

Another model based on the brane scenario is the ekpyrotic universe [241], the cyclic. version of which shall be considered in Section 10.⁶³

7. Variable cosmological constant

General Relativity allows for the introduction of only one arbitrary constant, the so-called cosmological constant Λ . At least two attitudes can be taken regarding Λ [333]. The first one is to consider it as a derived quantity, that emerges from vacuum fluctuations (see for instance [424]). One way out of the huge disagreement between theory and observation in this case [92] is to assume that Λ is actually time-dependent. The second attitude that can be adopted is that Λ is, along with *G*, a fundamental parameter of the theory, to be determined by observation⁶⁴ [323]. In fact, from a gravitational point of view what matters is the "effective" cosmological constant, since the matter Lagrangian can sometimes contribute with a Λ -like term, as in the case of the scalar field in the presence of a potential with a minimum:

 $\Lambda_{\rm eff} = \Lambda + V(\phi_{\rm min}),$

⁶² Further effects of angular momentum on the motion of the brane, including cyclic universes, were studied in [143].

⁶³ See [114] for an additional bouncing model using orientifolds.

⁶⁴ Notice that this second attitude is somewhat different from Einstein's original ideas leading to GR, since there would be curvature even in the absence of matter, caused by *A*.

where Λ is the "bare" cosmological constant. Any change in ϕ_{\min} during the evolution leads to changes in the value of Λ_{eff} . In fact, the effect of the evolution of the universe on the ground state is to add a temperature dependence, which can be translated into a time dependence [244]. A model along these lines based on a gauge field (instead of a scalar field) was presented in [322].⁶⁵ This is another motivation to consider a variable Λ , that is not a constant but a function of spacetime coordinates, in such a way that its value is determined by the dynamics of the theory under scrutiny (following the line of reasoning of other "variable constant" theories, see Section 10). In fact, a time-dependent cosmological constant has also been called upon to explain the current accelerated expansion and the fact that this phase started in the recent past.

In the case of $\Lambda = \Lambda(t)$, EE for the FLRW metric take the form

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\rho + \frac{\Lambda(t)}{3} - \frac{\epsilon}{a^2},$$
(235)

 $\ddot{a} = \Lambda(t) = 1$

$$\frac{a}{a} = \frac{A(0)}{3} - \frac{1}{6}(\rho + 3p), \tag{236}$$

and the continuity equation is given by

$$\dot{\rho} + 3\frac{a}{a}\left(\rho + p\right) = -\dot{\Lambda}.$$
(237)

As seen from Eq. (237), Λ can supply or absorb energy from ordinary matter and radiation. In fact, it follows from this equation that

$$TdS = -Vd\Lambda. \tag{238}$$

Hence, Λ is a source of entropy. Requiring that dS/dt > 0 implies $d\Lambda/da < 0$ through cosmic expansion. Assuming that only radiation is present, Eq. (237) gives

$$\frac{\mathrm{d}\rho}{\mathrm{d}a} + \frac{\mathrm{d}\Lambda}{\mathrm{d}a} + \frac{4\rho}{a} = 0,$$

which can be integrated to

÷

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^4 - \frac{1}{a^4} \int_{a_0}^a A^4 \frac{dA}{dA} dA,$$
(239)

where $\rho = \rho_0$ when $a = a_0$, and the subindex 0 denotes quantities evaluated at t = 0. Notice that the model is completely determined in this case by providing the function $\Lambda = \Lambda(a)$, since Eq. (239) then yields $\rho = \rho(a)$, and a = a(t) follows from Eq. (235). A cosmological model based on this scenario was discussed in [331], where the dependence of Λ on a was fixed by imposing that $\rho = \rho_c$ for all values of t, where $\rho_c = 3H^2$ is the critical density. It follows from Eq. (235) that

$$\Lambda = \frac{\alpha \epsilon}{a^2}.$$
(240)

The conditions $\dot{A} \ge 0$ and $\dot{a} \ge 0$ give $\epsilon > 0$, hence $\epsilon = 1$. In the model presented in [331], at t = 0 the universe had only a nonzero cosmological constant. With $\rho_0 = 0$, Eqs. (239) and (240) give

$$\rho(a) = \frac{\alpha}{a^2} \left(1 - \frac{a_0^2}{a^2} \right). \tag{241}$$

Note that $\rho_0 = 0$ implies that $a_0 \neq 0$, in such a way that the singularity at t = 0 is absent. An estimation of a_0 was made in [331] by assuming that the maximum temperature reached is $T_{\text{max}} \sim M_{Pl}$, which gives

$$a_0 \sim \frac{2.5}{\sqrt{N}} \times 10^{-20} (\text{GeV})^{-1},$$

where N = N(T) is the effective number of degrees of freedom at temperature *T*.

The fact that this model does not display a horizon problem was also shown in [331]. In fact, the time t_c at which global causality is established is given by

$$t_c = a_0 \sinh \frac{\pi}{2} \sim 2.3 a_0$$

which indicates that global causal connection was established at a very early time. The model is also free of the monopole problem, but it is worth noting that there is an inflationary period. From Eq. (235) we get

$$a^2 = a_0^2 + t^2. (242)$$

⁶⁵ In fact, any classical nonlinear field theory (such as nonlinear electromagnetism) admits a fundamental state that generates a cosmological constant [324].

A peculiarity of this model is that $a \to \infty$ for $t \to \infty$, even though $\epsilon = 1$. Needless to say, other choices of Λ would give a different asymptotic behavior.

The same form of Λ , namely

$$\Lambda(t) = \frac{\gamma}{a(t)^2},\tag{243}$$

where γ is a constant to be determined by observation, was studied in [414], but without the assumption that $\rho = \rho_c$. The conservation equation (237) can be solved for dust and radiation. Inserting the solution in Eqs. (235) and (236) we get

$$\begin{split} & \frac{\dot{a}^2}{a^2} + \frac{\Upsilon}{a^2} = \frac{1}{3}\rho^{(i)}, \\ & \frac{\ddot{a}}{a} = -\frac{1}{6}\rho^{(i)}, \end{split}$$

where $\Upsilon = \epsilon - 2\gamma/3$ for radiation, and $\Upsilon = \epsilon - \gamma$ for dust, and $\rho^{(i)}$ is the energy density of dust or radiation for the case $\Lambda = 0$. These equations show that the effect of assuming that $\Lambda \propto a^{-2}$ is to shift the curvature parameter ϵ by a constant value. A nonsingular cosmological model based on the model presented in [414] has been analyzed in [2]. Notice that Eq. (243) along with condition $d\Lambda/da < 0$ require that γ be positive. A positive Λ for all t implies, through Eq. (236) that there may be a zero in \dot{a} , and hence the possibility of a bounce. For this to happen we need that \dot{a} be zero at the putative bounce. Supposing there is a bounce, it follows from Eq. (235) evaluated at the bounce that

$$\alpha^{-1}\rho_0 a_0^2 = \epsilon - \gamma.$$

Hence, $\rho_0 > 0$ implies that $\epsilon > \gamma > 0$, and so $\epsilon = 1$. Introducing the *Ansatz* (243) in the Friedmann equation, we get

$$a^{2}\dot{a}^{2} = (2\gamma - 1)(a^{2} - a_{0}^{2}), \tag{244}$$

so it follows that $\gamma > 1/2$. Hence, $1/2 < \gamma \le 1$. This equation can be integrated to get

$$a^2 = (2\gamma - 1)t^2 + a_0^2,$$

which leads to bounded-from-above densities and temperature.⁶⁶

Yet another form for the dependence of Λ , given by

$$\Lambda = \Lambda_1 + \Lambda_2 \ a^{-m},$$

where Λ_1 , Λ_2 and *m* are constants (with $\Lambda_2 > 0$), was studied in [283]. The analysis of the dynamics was carried out using the analog of the one-dimensional problem of the particle under the influence of the potential V(a) given by

$$V(a) = -\Lambda_1 \delta \frac{a^2}{\alpha + 2} - \Lambda_2 \delta \frac{a^{2-m}}{\alpha - m + 2} + ba^{-\alpha}$$

where $\alpha = 1 + 3\lambda$, $\delta = 1 + \lambda$, *b* is a positive integration constant, and $p = \lambda \rho$. Denoting by *r* the maximum of the potential, cyclic solutions are obtained for the cases $\epsilon = 1$ with Λ_1 , $\Lambda_2 > 0$, and r > -1, and for $\Lambda_1 < 0$, $\Lambda_2 > 0$, and $m \le 2$, regardless of the sign of ϵ .

The proposal in Eq. (243) was later generalized in [21] to

$$\Lambda = 3\beta H^2 + \frac{3\gamma}{a^2},\tag{245}$$

where β and γ are dimensionless numbers, and $H = \dot{a}/a$ following [101]. With this *Ansatz*, the Friedmann equation for a radiation-dominated phase can be rewritten as

$$\dot{a}^2 = \frac{2\gamma - \epsilon}{1 - 2\beta} + A_0 a^{-2 + 4\beta},$$
(246)

which allows a bouncing solution at t = 0 for $A_0 < 0$, $\beta < 1/2$, $\epsilon = 1$ (with $\rho_0 > 0$). The value $\gamma = 1$ was chosen in [21] so that dS/da is always greater than zero, thus solving the entropy problem. In this case, the model gives $\Omega < 1$ for all t.

A thorough review of variable-*A* models has been presented in [330]. The models analyzed were power-laws of the different relevant parameters, namely

$$\Lambda_1 = \mathcal{A}t^{-\ell}, \qquad \Lambda_2 = \mathcal{B}a^{-m}, \qquad \Lambda_3 = \mathcal{C}H^n, \qquad \Lambda_4 = \mathcal{D}q^r,$$

where \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} , ℓ , m, n, and r are constants. Let us state from [330] the relevant results for this review: (1) no bouncing models were found for Λ_1 with k = 0 and $\ell = 1, 2, 3, 4$, irrespectively of the sign of \mathcal{A} . (2) For Λ_2 , it was shown (numerically) that there are nonsingular models for dust, $\epsilon = 1$, with m = 1, $\Omega_0 = 0.34$, and $0.68 < \Omega_{0A} < 0.72$, and also with higher

⁶⁶ The evolution of perturbations in this model was studied in [1].

values of *m* and Ω_0 . (3) For Λ_3 , the value n = 2 admit analytical solution. For this *n*, there are bouncing solutions for $\gamma > 2/3$ and $\epsilon = 1$ with $\mathcal{C} > 3(3\gamma/2 - 1)\Omega_0$, and also for $\gamma > 2/3$ and $\epsilon = -1$, for $\mathcal{C} < 3(3\gamma/2 - 1)\Omega_0$. (4) Only the value r = 1 was explored for Λ_4 . Defining $\lambda_0 = -\mathfrak{D}q_0/3$, there are closed bouncing solutions for $\lambda_0 > -\Omega_0$, and open bouncing solutions for $\lambda_0 < -\Omega_0$.

The examples given above show that varying- Λ scenarios are worth examining because they address a number of pressing problems in cosmology (horizon problem, entropy, initial singularity).⁶⁷ Furthermore, many of them are simple enough to draw definite conclusions about their viability. One of the drawbacks is perhaps the lack of strong motivation for choosing any given form of Λ . In this regard, let us remember that many of the varying- Λ models can be reverse-engineered to scalar-field models with a potential. Unfortunately, in most cases the corresponding models lack predictive power or clear particle physics motivation [333].

8. Past-eternal universes

In this section, we shall examine some models which are nonsingular but do not exhibit a bounce. Historically, perhaps the most important example of these is the Steady-State model [68].⁶⁸ As mentioned in Section 2.1, nonsingular solutions that start from a deSitter state were discussed in [384,164]. Another example is that discused in [292] in which every contracting and spatially flat, isotropic universe avoids the big crunch by ending up in a deSitter state enforced by the limiting curvature hypothesis.

8.1. Variable cosmological constant

As noted in [260], in all the articles mentioned in Section 7, the dependence of Λ on a and \dot{a} was set either from "first principles" (for instance quantum gravity, as in [414]), or by extrapolating backwards current cosmological data, including the current value of Λ . However, another view can be taken. Since Λ can be considered as a remnant of a period of inflation, a complete model should also describe the era of inflationary expansion. This is precisely the proposal in [260], where Λ was taken as

$$\Lambda(H) = 3\beta H^2 + 3(1-\beta)\frac{H^3}{H_{\ell}},$$
(247)

where H_{ℓ} is the timescale of inflation, and β is a parameter. Note that when $H = H_{\ell}$, $\Lambda = 3H_{\ell}^2$, as required by inflation, while $\Lambda \sim 3\beta H^2$ for large cosmological times. In the case of $\epsilon = 0$, and for

 $p = (\gamma - 1)\rho,$

an equation for the Hubble parameter follows [260]:

$$\dot{H} + \frac{3\gamma(1-\beta)}{2} H^2 \left(1 - \frac{H}{H_\ell}\right) = 0,$$

whose solution is

$$H = \frac{H_\ell}{1 + Ca^{3\gamma(1-\beta)/2}},$$

where C is a γ -dependent integration constant.⁶⁹ This equation can be integrated to yield

$$H_{\ell}t = \ln\left(\frac{a}{a_*}\right) + \frac{2C}{3\gamma(1-\beta)} a^{3\gamma(1-\beta)/2},$$

where a_* is an arbitrary value of the scale factor. It follows from this equation that the evolution of the universe starts from a deSitter stage $a \sim e^{H_\ell t}$ for $Ca^{3\gamma(1-\beta)/2} \ll 1$, and evolves towards a FLRW phase, $a \sim t^{2/3\gamma(1-\beta)}$ for $Ca^{3\gamma(1-\beta)/2} \gg 1$.

8.2. Fundamental state for f(R) theories

A novelty in some theories described by Lagrangians that depend only on *R* is the possibility of the emergence of an intrinsic cosmological constant. This is not the case, however, in theories generated by Lagrangians that are a linear combination of R^2 and $R_{\mu\nu}R^{\mu\nu}$ as can be seen by a direct inspection of the EOM (35). The proof of this assertion follows from the fact that the tensors $\chi_{\mu\nu}$ and $Z_{\mu\nu}$ appearing in the EOM (27) are traceless in the case of a constant curvature scalar

⁶⁷ Nonsingular cosmological solutions for the case in which the cosmological constant is replaced by a second-rank tensor Λ^{μ}_{ν} were studied in [81].

⁶⁸ For an updated version, see Section 10.2.5.

⁶⁹ Here the value $\epsilon = 0$ was chosen, but this restriction was lifted in [261].

 $(R_{\mu\nu} = \Lambda g_{\mu\nu})$. However, restricting to the f(R) case, Lagrangians that are not linear in R^2 can bypass such prohibition. The existence of a deSitter solution [130] in the absence of matter occurs when the function obeys the condition

$$\frac{f'}{f} = \text{constant.}$$
(248)

A typical example is provided by the exponential Lagrangian

$$f(R) = \exp\left(\frac{R}{2\Lambda}\right).$$

It follows straightforwardly from Eq. (36) that $R_{\mu\nu} = \Lambda g_{\mu\nu}$ is a possible state of the system.

8.3. The emergent universe

Another example of past eternal universe was given in [151]. This model uses general relativity plus a scalar field with a potential, and matter. The relevant equations are

$$\begin{split} \phi &+ 3H\phi + V'(\phi) = 0, \\ \frac{\ddot{a}}{a} &= -\left[\frac{1}{2}(1+3\omega)\rho + \dot{\phi}^2 - V(\phi)\right] \\ H^2 &= \rho + \frac{1}{2}\dot{\phi} + V(\phi) - \frac{\epsilon}{a^2}. \end{split}$$

From these, it follows that to have a minimum of the scale factor we need to impose the conditions

$$\frac{1}{2}(1+3\omega)\rho + \dot{\phi}^2 < V(\phi),$$

and

$$\frac{1}{2}\dot{\phi}_i^2 + V_i + \rho_i = \frac{\epsilon}{a_i^2},$$

where the subindex *i* means that the quantities are evaluated at t_i , the time at which *a* is minimum. Assuming positive potentials and energy density, it follows that only $\epsilon = +1$ is allowed. It follows that

$$\frac{1}{2}(1-\omega_i)\rho_i+V_i=\frac{2}{a_i^2},$$

where $V_i = \Lambda_i$, and

$$(1+\omega_i)\rho_i+\dot{\phi}^2=\frac{2}{a_i^2},$$

so a model can be constructed with $\rho_i = 0$ and constant $\dot{\phi}^2$. This can be achieved in the limit $t \to \infty$ with the potential [151]

$$V(\phi) = V_f + (V_i - V_f) \left[\exp\left(\frac{\phi - \phi_f}{\alpha}\right) - 1 \right]^2,$$

where ϕ_f is the value of the field for which *V* is minimum, and α is a constant energy scale. In order to achieve the Einstein universe state in the far past, some fine-tuning on a_i and $\dot{\phi}_i$ is needed, which is not necessarily a hindrance [151].⁷⁰ The choice of such a highly-symmetric state as the initial state is supported by various arguments: it is stable against some types of inhomogeneous linear perturbations,⁷¹ it has no horizon problem, it maximizes the entropy within the family of FLRW radiation models, and it is the unique highest symmetry non-empty FLRW model (with a 7 dimensional group of isometries). The model was elaborated further in [153], where it was shown that an explicit form for the potential can be found such that the model leaves the inflationary stage and enters a reheating phase, followed by standard evolution.

 $^{^{70}}$ In particular, the initial scale factor could be chosen in such a way to avoid the quantum gravity regime.

⁷¹ But notice that it is not stable under homogeneous perturbations.

9. Quantum cosmology

As discussed in Section 1, there are reasons to suppose that at very high energies some of the hypotheses of the singularity theorems are rendered invalid: if the universe ever attains this regime, an important role is to be played by quantum gravitational effects, in such a way that a quantum theory of gravitation is needed to have a proper description.

Although there is yet no complete realization of quantum gravity, there are some attempts to tackle the singularity problem in a quantum framework. A standard method of quantizing General Relativity is canonical quantization [199] where the momentum and Hamiltonian constraint equations are interpreted as operators, and it is required that they annihilate the quantum state. The Hamiltonian constraint gives the Wheeler–DeWitt (WdW) equation [416], which depends on the choice of the factor ordering in the products of generalized momenta and "velocities". For some choices of ordering, the WdW equation turns it into a Klein–Gordon equation on an indefinite DeWitt metric in infinite-dimensional superspace (space of three-metrics), with a potential term [416]. In addition to the WdW equation, initial conditions must be specified, the two most popular being the "no-boundary" [203], and the "tunnelling" condition [409].

In practice, the infinite degrees of freedom of superspace are truncated to obtain a minisuperspace model, usually under the assumptions of isotropy and homogeneity. Once a solution to the WdW equation has been found, there is the question of how to interpret it and extract probabilities from it.

Among other issues related to the WdW equation, there is the fact that a suitable initial condition must be chosen to get a solution. It would be desirable that the initial condition be somehow determined by the dynamical law (see for instance [62]). In fact, the most well-accepted proposals mentioned above do not solve the singularity problem [23]. Moreover, in the quantization following the ADM procedure, time is fixed by a gauge choice, and the results are dependent of this choice [339].⁷²

As we shall see below, there are other approaches to Quantum Cosmology which may yield a nonsingular universe in the regime where the WdW equation is valid. We shall discuss two possibilities: the Bohm-de Broglie interpretation of QM, and Loop Quantum Cosmology (LQC).

9.1. The ontological (Bohm-de Broglie) interpretation

If the universality of quantum mechanics is assumed, the Universe must be describable by a wave function (furnished by a yet-to-be-discovered quantum theory of gravity and matter fields) in every step of its evolution. Moreover, this description must have a well-defined classical limit. The orthodox interpretation of Quantum Mechanics (the so-called Copenhagen interpretation) [221] is ill-suited for the task of describing the universe, since it assumes the existence of a "classical apparatus" external to the system to solve the measure problem by forcing the collapse of the wave function. Clearly, there is no classical apparatus outside the universe. Therefore, the least we can say is that an alternative to the Copenhagen interpretation is needed. One such alternative that has received some attention recently is that of Bohm and de Broglie (BdB)[59].⁷³ In classical physics, the dynamics of a point in configuration space is determined by the principle of extremal action, yielding the classical EOM. According to the BdB interpretation, in quantum physics the evolution of the configuration variables is guided by a quantum wave which obeys Schrödinger's equation. The associated Hamilton–Jacobi equation displays a new term (of quantum origin, see below), that can be interpreted as part of the potential. It should be emphasized that the BdB interpretation furnishes a framework to make predictions based on the wave function of the system, which must be obtained by some means (for instance, through the WdW equation).

Let us briefly review first the quantum mechanics of a single particle in the BdB interpretation, and afterwards the results will be translated, *mutatis mutandis*, to the context of FLRW cosmology. The Schrödinger equation for a non-relativistic particle in a potential *V* is given by

$$i\hbar \frac{\mathrm{d}\psi(x,t)}{\mathrm{d}t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi(x,t).$$

With the replacement $\psi = R \exp(iS/\hbar)$, this equation becomes

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0,$$
(249)
$$\frac{\partial R}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m}\right) = 0.$$
(250)

This last equation suggests that $\nabla S/m$ can be interpreted as a velocity field, leading to the identification $p = \nabla S$, in such a way that Eq. (249) is the Hamilton–Jacobi equation for the particle in the classical potential *V* plus a "quantum potential"

⁷² In this regard, it was shown in [126] that a Bianchi I universe, quantized following the ADM recipe with a particular choice of the time coordinate [269] in the presence of dust is nonsingular.

⁷³Other possibilities (not free of problems, though) are the many-worlds interpretation [156], non-linear quantum mechanics [182], and decoherence [191].

 $Q = -\hbar^2 \nabla^2 R/2mR$. The BdB interpretation argues that a quantum system is composed of a particle *and* a field, and that quantum particles follow trajectories x(t), independent on the existence of an outside observer. These trajectories can be determined from

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\nabla V - \nabla Q,$$

or from $p = m\dot{x} = \nabla S$, after *S* and *R* are determined using Eqs. (249) and (250). In practice, since *S* is the phase of the wave function, it can be read off from the explicit solution of Schrödinger's equation.

Let us analyze an example developed in [110], where the Lagrangian was given by

$$\mathcal{L} = \sqrt{-g} \left(R - C_{\omega} \phi_{,\mu} \phi^{,\mu} \right),$$

where $C_{\omega} = (\omega + \frac{3}{2})$. From the metric

$$ds^{2} = -N^{3}dt^{2} + \frac{a(t)^{2}}{1 + (\epsilon/4)r^{2}} \left(dr^{2} + r^{2}d\Omega^{2}\right)$$

and the definitions $\beta^2 = 4\pi \ell_{\rm Pl}^2/3V$, $\bar{\phi} = \phi \sqrt{C_\omega/6}$, we get

$$\mathcal{H} = N\left(-\beta^2 \frac{p_a^2}{2a} + \beta^2 \frac{p_{\phi}^2}{2a^3} - \epsilon \frac{a}{2\beta^2}\right),$$

with $p_a = -a\dot{a}/(\beta^2 N)$, $p_{\bar{\phi}} = a^3 \dot{\bar{\phi}}/(\beta^2 N)$. Defining $\tilde{a} = a/\beta$, setting $\beta = 1$ and $\alpha \equiv \ln \tilde{a}$, we get

$$\mathcal{H} = \frac{N}{2\exp(3\alpha)} \left(-p_{\alpha}^2 + p_{\phi}^2 - \epsilon \exp(4\alpha) \right), \tag{251}$$

where

$$p_{lpha} = -rac{\dot{lpha} \mathrm{e}^{3lpha}}{N}, \qquad p_{\phi} = rac{\dot{\phi} \mathrm{e}^{3lpha}}{N}.$$

Notice that $p_{\phi} = \bar{k}$ is a constant of the motion. We shall restrict to the case $\epsilon = 0$ since it is analytically tractable. The classical solutions are given by

$$a = 3\bar{k}t^{1/3}, \qquad \phi = \frac{1}{3}\ln t + c_2,$$

where c_2 is an integration constant. Depending on the sign of \bar{k} , this solution contracts to or expands from a singularity. The Wheeler–DeWitt equation corresponding to the Hamiltonian given in Eq. (251) is given by [110]

$$-\frac{\partial^2 \Psi}{\partial \alpha^2} + \frac{\partial^2 \Psi}{\partial \phi^2} + \epsilon \ \mathrm{e}^{4\alpha} \Psi = 0.$$

The solution, obtained by separation of variables, reads

$$\Psi(\alpha,\phi)=\int F(\kappa)A_{\kappa}(\alpha)B_{\kappa}(\phi)\mathrm{d}\kappa,$$

where κ is a separation constant, $F(\kappa)$ is an arbitrary function of κ ,

$$A_{\kappa}(\alpha) = a_1 \exp(i\kappa\alpha) + a_2 \exp(-i\kappa\alpha),$$

(for $\epsilon = 0$), and

 $B_{\kappa}(\phi) = b_1 \exp(i\kappa\phi) + b_2 \exp(-i\kappa\phi).$

A direct application of the formalism sketched for the case of a one-particle system, generalized to several degrees of freedom yields from the Hamiltonian (251) [110]

$$Q(\alpha, \phi) = \frac{e^{3\alpha}}{2R} \left(\frac{\partial^2 R}{\partial \alpha^2} - \frac{\partial^2 R}{\partial \phi^2} \right),$$

with the "guidance relations"

$$\frac{\partial S}{\partial \alpha} = -\frac{\mathrm{e}^{3\alpha} \dot{\alpha}}{N}, \qquad \frac{\partial S}{\partial \phi} = \frac{\mathrm{e}^{3\alpha} \dot{\phi}}{N}.$$

A state is now needed to read off from it S and R. A Gaussian superposition was chosen in [110], given by

$$\Psi(\alpha,\phi)=\int F_{\kappa}B_{\kappa}(\phi)[A_{\kappa}(\alpha)+A_{-\kappa}(\alpha)]\,\mathrm{d}\kappa,$$



Fig. 10. Field plot of the planar system (252)–(253) for $\sigma = d = 1$. Taken from [110].

with

 $\dot{\phi}$

$$F(\kappa) = \exp\left(-\frac{(\kappa-d)^2}{\sigma^2}\right)$$

and $a_2 = b_2 = 0$. Performing the integration in κ , we can extract from the result the phase *S* which, when inserted into the guidance relations (in the N = 1 gauge) furnishes a planar system:

$$\dot{\alpha} = \frac{\phi\sigma^2 \sin(2d\alpha) + 2d\sinh(\sigma^2 \alpha \phi)}{\exp 3\alpha \left(2(\cos(2d\alpha) + \cosh(\sigma^2 \alpha \phi))\right)},\tag{252}$$

$$=\frac{-\alpha\sigma^{2}\sin(2d\alpha)+2\alpha\cos(2d\alpha)+2\alpha\cos(\sigma^{2}\alpha\phi))}{\exp 3\alpha\left(2(\cos(2d\alpha)+\cosh(\sigma^{2}\alpha\phi))\right)}.$$
(253)

The plot of this system (see Fig. 10) shows that there are bouncing trajectories for $\alpha > 0$, and also oscillating universes near the centre points (white points in the plot). The BdB interpretation has been applied to mini-superspace models in Quantum Cosmology (see for instance [6,7,339]), and non-singular solutions have been found for models with scalar fields or radiation [157]. The bounce is due to the action of the quantum potential, which generates a repulsive "quantum force", large enough to reverse the collapse.

One of the advantages of this formulation is that, starting from WdW equation, it yields a dynamics that is invariant under time re-parameterizations. Notice however that the results are dependent on the state chosen to represent the system.

9.2. Loop quantum gravity

Loop Quantum Gravity is a background-independent, non-perturbative canonical quantization of gravity in which the classical metric and the extrinsic curvature are turned into operators on a Hilbert space [359]. The classical description of space-time is replaced by a quantum counterpart, in such a way that quantum effects are important at very short scales, for instance near putative singularities. In this scenario, the evolution of the universe is divided in three epochs. First there is a quantum epoch with high curvature and energy, described by difference equations for the wave function of the universe. These are a direct consequence of the discreteness of space and time, the step size being dictated by the lowest non-zero eigenvalue of the area operator (see [62]). It is this discreteness that modifies the behavior near the singularity, leading to a theory that is not equivalent to the WdW description (even in the isotropic case), which furnishes a continuous spectrum for the scale factor. A semiclassical epoch follows, with differential equations for matter and geometry modified by non-perturbative quantization effects. Finally, a classical phase is reached, described by the usual cosmological equations.

Since difference equations are often difficult to analyze or to solve explicitly, and at such a fundamental level, the emergence of space-time in inhomogeneous models with many degrees of freedom from the underlying quantum state is hard to understand, a suitable strategy is to use special models allowing exact solutions. Care must be taken in the extension of results from particular examples to more general cases [83]. In any case, it may be instructive to have a detailed understanding of how the singularity is resolved in some instances.

Yet another convenient simplification is to work in an effective semiclassical theory, which takes into account only some quantum effects. This theory can be understood as governing the motion of a wave packet that solves the difference equation [64], and can be obtained as an asymptotic series of correction terms to the equations of motion in the isotropic case [65]. For instance, in the case of a matter term generated by a scalar field under the influence of a potential, the effective Klein–Gordon equation is [377]

$$\ddot{\phi} = \dot{\phi} \left(-3H + \frac{\dot{D}}{D} \right) - DV'(\phi), \tag{254}$$

where

$$D(q) = \left(\frac{8}{77}\right)^6 q^{3/2} \{7[(q+1)^{11/4} - |q-1|^{11/4}] - 11q[(q+1)7/4] - |q-1|^{7/4} \operatorname{sgn}(q-1)\}^6,$$

with $q = a^2/a_*^2$ and $a_*^2 = \gamma \ell_{Pl}^2 j/3$, where $\gamma \approx 0.13$, and *j* is a quantization parameter, which takes half-integer values. This equation represents an approximate expression for the eigenvalues of the inverse volume operator [61]. The function *D* varies as a^{15} for $a \ll a_*$, has a global maximum at $a \approx a_*$, and falls monotonically to D = 1 for $a > a_*$. In turn, the effective Friedmann equation is given by

$$\frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2} = \frac{1}{3} \left(\frac{\dot{\phi}^2}{2D} + V(\phi) \right),\tag{255}$$

and the effective Raychaudhuri equation is

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\dot{\phi}^2 \left(1 - \frac{\dot{D}}{4HD}\right) + \frac{1}{3}V(\phi).$$
(256)

These approximations are valid for $a_i < a < a_*$, where $a_i = \sqrt{\gamma} \ell_{Pl}$. Below a_i the quantum nature of spacetime cannot be replaced by an effective theory, while above a_* we recover classical cosmology. It was shown in [377] that a closed universe with a minimally coupled scalar field will bounce (avoiding the so-called big crunch) as soon as $a \approx a_*$ for any choice of the initial conditions. The bounce in this case is due to the change of sign of the "friction" term in Eq. (254), which becomes frictional for $a \ll a_*$, freezing the field ϕ in some constant value, and turning the effective EOS into a cosmological constant EOS [377]. Similar results were obtained in the case of anisotropic models [63].

The previous example incorporated quantum gravitational effects on matter (represented by a scalar field) Hamiltonian, but there may also be modifications of the gravitational Hamiltonian due to quantum geometry. Recently, some calculations illustrating the effects of quantum geometry on both the gravitational and matter Hamiltonians were carried out in the case of a spatially homogeneous, isotropic $\epsilon = 0$ universe with a massless scalar field (a system which is singular both classically and according to the WdW formalism in the Copenhagen interpretation of QM). It was shown in [23] that the singularity is resolved in the sense that a complete set of Dirac observables on the physical Hilbert space remains well-defined throughout the evolution; the big-bang is replaced by a big-bounce in quantum theory due to quantum corrections to the geometry; there is a large classical universe on the "other side", and the evolution bridging the two classical branches is deterministic, thanks to background independence and non-perturbative methods.⁷⁴ Notice also that no boundary condition was imposed (it was asked instead that the quantum state be semiclassical at late times).⁷⁵

Surely the major limitation in all the analysis of LQC is that, since a satisfactory quantum gravity theory which can serve as an unambiguous starting point is not available yet, the theory is not developed by a systematic truncation of full quantum gravity. Another limitation is the restriction to isotropy and homogeneity.

9.3. Stochastic approach

A different approach was introduced in [314], which starts from the observation made in [277] that the universe could be enlarged through an "analytic extension". In [277], such an extension is achieved from the geometrical construction of a semiclosed universe, namely a closed Friedmann model extended by gluing a given geometry to the FLRW before the maximum expansion. This gluing can be done in different ways, through the junction conditions. In [277] an asymptotically flat geometry was chosen. A collection of this configuration (called *friedmon* in [314]) was considered in [314], in such a way that each member of the collection perceives the remaining systems as a perturbative effect of random character, as in a stochastic process. Noting that in the case of an open universe, the Friedman equation takes the form of energy conservation for a harmonic oscillator, namely

$$\dot{a}^2 + \frac{1}{3}\Lambda a^2 = 1.$$

a Hamiltonian can be defined by setting q = a, $p = \dot{a}$, and the quantum theory of the harmonic oscillator can be developed according to [304]. A straightforward calculation leads to the result

$$E[a^2(t,W)] = a_{Cl}^2 + \frac{1}{2}\sqrt{3}\frac{\hbar}{\Lambda},$$

where *E* is the expectation value, a_{Cl} is the classical value of *a*, and *W* is the white noise associated to the stochastic process. One arrives at the result that the net effect of the environment is to preclude the collapse of the model, the minimum of the radius being large if Λ is small.

⁷⁴ In a subsequent paper the Hamiltonian was modified to forbid the bounce at low densities [24].

 $^{^{75}}$ An analysis along the same lines was carried out in [404] for the case $\epsilon = -1$, and it was shown that the singularity is avoided too.

10. Cyclic universes

Oscillating universes have been explored in several contexts in an attempt to solve some problems of the standard cosmological model. The first example of such universes was that presented in the seminal paper by Lemaître [254], who stated that "The solutions where the universe successively expands and contracts, periodically reducing to an atomic system with the dimensions of the solar system, have an incontestable poetic charm, and bring to mind the Phoenix of the legend" [254].⁷⁶ Let us briefly recall some of the issues of the standard model and the solution that oscillating models can provide:

• The flatness problem. The Friedmann equation can be written as

$$|\Omega_{\rm tot}(t) - 1| = \frac{|\epsilon|}{a^2 H^2}$$

As already discussed in Section 1, in a situation in which the universe is dominated by matter or radiation, the difference $|\Omega_{tot}(t) - 1|$ grows as a power of *t*. Since present data indicate that Ω_{tot} is very close to 1, it must have been incredibly close to one far in the past, if $\Omega_{tot} \neq 1$ initially. This is the so-called flatness problem. As we shall see below, in a cyclic universe Ω_{tot} starts deviating from 1 only when *a* approaches its maximum. Since the maximum grows with the number of cycles, in a sufficiently old cyclic universe it may take a long time for Ω_{tot} to deviate from 1 [138].

• The horizon problem. In the SCM, light signals can propagate only a finite distance between the initial singularity and a given time *t*, provided the energy density changes faster than a^{-2} . Hence, microphysics would not have enough time to take the universe to its high degree of homogeneity. In the cyclic model the age of the universe is given by the sum of the duration of all the previous cycles. This would solve the horizon problem, provided correlations safely traverse the bounce.

Some implementations of the cyclic model may also solve the so-called "coincidence problem" (why did the universe begin its accelerated expansion only recently?). The model in [419] has its parameters tuned in such a way that the fraction of time that the universe spends in the coincidence state is comparable to the period of the oscillating universe.

Oscillating models have been also used to explain observed values of the dimensionless constants of nature. In [379], the value of these constants is randomly set after a bounce (see also [290]). In order to see whether cosmological evolution establishes any trend in the behaviour of the "constants", cyclic models were studied in [37] as solutions of varying-constants theories, such as the varying α theory presented in [366], the Brans–Dicke theory, and the variable-speed-of-light theory [275].⁷⁷ Cyclic solutions were studied both for non-interacting and interacting scalar field (which models evolution of the "constant") plus radiation, and the bounce was caused by negative-energy scalar fields. In all three theories, the models showed monotonic changes in the constants from cycle to cycle (the scale factor qualitatively behaving as explained in [35]).

10.1. Thermodynamical arguments

The existence of oscillatory solutions in the FLRW model was shown by Tolman (see [395] and references therein, and [202,350]). His argument can be understood from a purely mechanic point of view, by modelling the Friedmann equation as a one-particle system: examination of the effective potential for a closed universe shows that there are oscillatory solutions for some values of the parameters of the model (assuming that there is a mechanism to revert the contraction into expansion before the singularity). These solutions are permitted from a thermodynamical point of view, since the matter term in the FLRW model is a perfect fluid, whose entropy is constant. Hence the expansion is reversible, although at a finite rate. In more realistic models however, entropy generation is inevitable, arising from various sources (such as viscosity effects from particle creation) [183]. However notice that, as discussed in [395], the entropy of each element of the fluid need not attain a maximum, as would be the case in an isolated thermodynamical system, because the energy of the fluid element is not constant. In fact, each time a given element of fluid returns to the same volume, its energy density is higher than in the previous passage through the same volume, due to a lag behind equilibrium conditions. The increment in entropy leads to non-reversibility, which forbids identical oscillations. As a consequence of the raising energy density, the maximum value for *a* grows in each cycle.⁷⁸ This can be easily seen from Friedmann equation, taking the case $\Lambda = 0$, $\epsilon = 1$ as an example:

$$\dot{a}^2 + 1 = \frac{1}{3}\rho a^2.$$

After one cycle, the 3-volume goes back to a value it had before when *a* does. Since ρ grows with the number of cycles, this growth can only be attributed to an increment in *a*. Hence a sufficiently "old" cycle is strongly peaked, and Ω_{tot} remains close to 1 until *a* is very near the maximum, thus yielding a solution to the flatness problem.

⁷⁶ Note however that Lemaître did not produce an explicit solution for the cyclic universe.

⁷⁷ A word of caution regarding this latter type of theory was issued in [152].

⁷⁸ Notice that in these considerations neither the mechanism that allows safe passage through the singularity nor the details of the entropy generation are given.



Fig. 11. The plot shows that transition of a $\Lambda > 0$, radiation-filled universe from the oscillating phase to the ever-expanding phase, due to the growth of entropy (given by the increment in $\lambda_n = \sqrt{4C_m \Lambda/3}$). Taken from [35].

Starting from the fact that the entropy of the universe today is finite, and making the reasonable hypothesis that the increment in the entropy through each bounce shares this property, Zeldovich and Novikov [426] among others (see [133]) have estimated the number of cycles back to an initial state (which should not be singular, to keep the idea of a cyclic universe attractive).

To move from qualitative arguments to actual calculations, the key issue is the production of entropy. Irreversible energy transfer from the gravitational field to particle generation was the source of entropy considered in [346], while it was suggested in [193,124] that black hole evaporation could be responsible for entropy growth. An analytical study that showed the correctness of Tolman's arguments was presented in [35], where closed Friedmann universes with $\Lambda \neq 0$ were scrutinized, including an *ad-hoc* mechanism of entropy generation, and assuming that there is a bounce, without entering in the details of its realization. The entropy growth was implemented by relating the constant coming from the conservation laws

$$\rho_i a^{\alpha} = \text{constant} = C_i,$$

where *i* denotes radiation or dust, and $\alpha = 4$ or 3 respectively, to the expression for the entropy in each case. Let us take the case of radiation, in which

$$S_r = \text{constant} = \frac{8}{3}\pi^2\beta T^3a^3,$$

so we can set $T^3a^3 = \text{const.} = \gamma$. From this equation and the conservation law it follows that

$$C_r = \frac{G\gamma^{1/3}}{\pi c^4} S_r,$$

thus linking the increment in entropy to the change in the constant appearing in the solution. In the same way it is shown that C_m is related to S_m through a similar expression. In [35] it was assumed that the entropy is constant within a cycle, but increases at the beginning of each cycle through the increment in the constants C_r and C_m . The behaviour of models with different combinations of matter, radiation and cosmological constant were studied for positive and negative Λ . The results show that for $\Lambda > \Lambda_c$ (where $\Lambda_c = \Lambda_c(C_r, C_m)$) the universe stops its oscillations with increasing maximum and starts an ever-expanding phase (see Fig. 11). In other words, when the oscillations become large enough the cosmological constant dominates over the matter and radiation terms, the oscillations cease, and the universe enters a deSitter regime. If $\Lambda < \Lambda_c$, the oscillations are not interrupted. Oscillations in anisotropic models were also studied in [35], paying attention to the question of isotropization after a large number of oscillations. As the entropy increases, the volume of Bianchi I universes with $\Lambda < 0$ oscillates with growing maximum amplitude, while shear anisotropy vanishes.⁷⁹

A more sophisticated model was studied in [106], where FLRW two-fluid out of equilibrium models were considered. Exact solutions were found for a particular cases of the energy exchange, conserving the total energy. In the case of nonzero spatial curvature, cyclic models were shown to exist. The energy exchange between the fluids was modelled by a function *s* such that

$$\dot{\rho} + 3H\gamma\rho = s, \qquad \dot{\rho}_1 + 3H\Gamma\rho_1 = -s,$$

where $\gamma - 1$ and $\Gamma - 1$ are the EOS parameters of each fluid. Solutions of these equations along with

$$H^2 = \rho + \rho_1 - \frac{\epsilon}{a^2}$$

⁷⁹ Axisymmetric Bianchi type IX, dust Kantowski-Sachs, Bianchi IX, and some features of inhomogeneous cyclic cosmological models were also studied in [35].

were found in [106] for different forms of *s*, for the cases radiation and dust, radiation and scalar field, and radiation and negative vacuum energy. In the second case, a new feature appears (as well as the "runaway stage" mentioned in [35]): the increment in magnitude of the minima in the scale factor as time increases. This was interpreted by the authors as a consequence of the energy exchange: the scalar field reached negative energy values after transferring energy to radiation. Surely this behaviour depends on the specific form of the function *s*. The examples studied in [106] suggest that caution is needed when it is said that cyclic models can solve the flatness problem, since in some of them the cycles cannot become indefinitely large and long-lived, while in others the minimum of the expansion increases.

10.2. Realizations of the cyclic universe

We present in this section some concrete examples of theories that yield cyclic regular solutions (i.e. which actually bounce at the minimum of the expansion without presenting singularities), along with some of its successes and conundrums.

10.2.1. Changes in the matter side of EE

One way to generate a cyclic universe is to add matter that will certainly produce a bounce, and consider next what conditions are to be imposed on it to produce oscillations. A necessary condition that the extrema of the expansion factor must satisfy is given by H = 0, with

$$H^{2} = \frac{8\pi}{3M_{\rm Pl}^{2}}(\rho - f(\rho)).$$

This amounts to $\rho - f(\rho) = 0$, where the function $f(\rho)$ is positive. A cyclic universe has been generated along this line in [119,234], where "wall-like" and "string-like" matter (whose energy scales as a^{-1} and a^{-2} respectively) generate the required $f(\rho)$.⁸⁰ These rather exotic sources can be also thought as originating from scalar fields under the influence of a potential, using the procedure presented in [33]. A modification of the Friedmann equation coming from brane models was used to fix the form of $f(\rho)$ in [82], where

$$H^{2} = \frac{8\pi}{3M_{p}^{2}} \left(\rho - \frac{\rho^{2}}{2|\sigma|}\right),\tag{257}$$

see Section 6. The dominant component in this model is the so-called "phantom" matter, which has an energy-conditions-violating equation of state characterized by

$$\omega_{\rm Q} = \frac{p_{\rm Q}}{\rho_{\rm Q}} < -1.$$

Since the energy density of matter with state parameter ω scales with the expansion as

$$\rho = a^{-3(1+\omega)},$$

we see that ρ grows with the expansion. Surely before reaching an infinite energy density, quantum gravity effects will take over the evolution. The somewhat paradoxical situation arises in which very high-density effects must be incorporated in the description of the universe for both very small and very large values of the scale factor. The central idea in [82] is that the same physics causes then the bounce and the turnaround, both governed by Eq. (257). After a bounce, the universe follows standard evolution until phantom energy dominates. This energy may erase every trace of structure [92], and dominates the evolution until high-density effects are again important, producing the turnaround. As will be discussed in Section 10.3, one of the problems to be faced in the collapsing phase is the merging of black holes into a "monster black hole". The energy density the universe must reach in order that black holes are torn apart was shown in [82] to be

$$ho_{\mathrm{br}} \propto M_{\mathrm{P}}^4 \left(\frac{M_{\mathrm{P}}}{M}\right)^2 \frac{3}{32\pi} \frac{1}{|1+3\omega_{\mathrm{Q}}|}.$$

This energy density must be reached before the turnaround, characterized by $\rho_{ta} = 2|\sigma|$. The value $\sigma \approx m_{GUT}$ is enough for all but Planck-mass black holes to be torn apart (some of them evaporate before the universe enters the phantom energy stage). These Planck-mass remnants may help in explaining the dark matter puzzle [82].⁸¹ Some problems still remain in this model. First, the generation of structures in the contracting phase needs to be addressed, to see that the black hole problem does not recur. Second, as stated before, entropy production would lead actually to quasi-cyclic evolution.

A similar model has been studied in [382], given by

$$H^{2} = \frac{1}{3}\rho + \nu\rho^{2} + \frac{\Lambda}{3},$$
(258)

⁸⁰ Earlier attempts along these lines, imposing that $p \propto -a^{-n}$, and $\rho = p \propto -a^{-6}$ are respectively given in [332,357]. For a somewhat different approach, see [213,423].

⁸¹ Details about the evolution of this model and its relation with the so-called coincidence problem can be found in [419].

where ν is a real constant. Analytical solutions of this equation have been found in the case of dust, and their generic feature seems to be the replacement of the initial singularity by a bounce, some solutions displaying also a cyclic behaviour (those for $\Lambda \leq 0$ and $\nu < 0$).

An interesting twist to the entropy problem in cyclic universes was introduced in [43], where a model described by Eq. (258) was studied, with the cosmological constant replaced by a dark energy component with EOS $p = \omega \rho$ and $\omega < -1$, matter and radiation as normal components, and $\nu < 0$. The model takes advantage of the Big Rip phenomenon, where bound systems become unbound and their constituents causally disconnected as a result of the increasing value of the dark energy density. As a consequence of the Big Rip, the universe would disintegrate in a huge number of disconnected patches. The new ingredient of the model is that the turnaround is placed an instant before the "total Big Rip", when each patch would contain almost no matter at all, and only a small amount of radiation [44] and dark energy. Due to the Big Rip, the huge entropy of the universe is distributed between the enormous number of patches, hence leaving each patch with very low entropy. The subsequent contraction of each patch is free of "formation of structure" problems, and proceeds until a bounce occurs. After the bounce, a normal inflationary phase follows (vastly increasing the entropy), and the cycle starts again.

10.2.2. Cyclic universes in nonlinear electrodynamics

As discussed in Section 4.5, nonlinear electrodynamics can describe a nonsingular universe. Here it will be shown how a cyclic model arises from the theory given by the Lagrangian [327]

$$\mathcal{L} = -\frac{1}{4}F + \alpha F^2 - \frac{\gamma^2}{F}$$
(259)

where α and γ are constants, with the dependence of the magnetic field on the scale factor given by $\mathscr{H} = \mathscr{H}_0/a^2$ (see Eq. (208)). The time-evolution of the scale factor can be qualitatively described by the effective potential, which arises from Friedmann equation written as a "one-particle" system. For the case at hand, the effective potential is given by

$$V(a) = \frac{A}{a^6} - \frac{B}{a^2} - Ca^6.$$
 (260)

The constants in V(a) are given by

$$A = 4\alpha \mathcal{H}_0^4, \qquad B = \frac{1}{6} \mathcal{H}_0^2, \qquad C = \frac{\gamma^2}{2\mathcal{H}_0^4}$$

and are all positive. The analysis of V(a) and its derivatives implies solving polynomial equations in *a*, which can be reduced to cubic equations through the substitution $z = a^4$. The existence and features of the roots of such equations are discussed in [55]. A key point to the analysis is the sign of *D*, defined as follows. For a general cubic equation

$$x^3 + px = q,$$

the discriminant D is given by

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

We will denote by D_V the discriminant corresponding to the potential and $D_{V'}$ that of the derivative of V. From the behaviour of the potential and its derivatives for $a \to 0$ and $a \to \infty$ we see that only one or three zeros of the potential are allowed. The case of interest here (given by $D_V > 0$, $D_{V'} = 0$) is plotted in Fig. 12, which shows the qualitative behavior of the potential for typical values of the parameters. The model is nonsingular for any value of ϵ , and a cyclic model is obtained for $\epsilon = 1$.

This setting was generalized in [328], where the Lagrangian

$$\mathcal{L}_T = \alpha^2 F^2 - \frac{1}{4}F - \frac{\mu^2}{F} + \frac{\beta^2}{F^2}$$
(261)

was considered, with α , β and μ constants. As shown in [328], four distinct phases can be described with this Lagrangian: a bounce, a radiation era, an acceleration era and a turnaround. This unity of four stages, christened *tetraktys* in [328], constitutes an eternal cyclic configuration. The cyclic behavior is a manifestation of the invariance under the dual map of the scale factor $a(t) \rightarrow 1/a(t)$, a consequence of the corresponding inverse symmetry of the Lagrangian (261) w.r.t. the electromagnetic field ($F \rightarrow 1/F$, where $F \equiv F^{\mu\nu}F_{\mu\nu}$). Restricting to a magnetic universe, as defined in Section 4.5.1, the Lagrangian \mathcal{L}_T yields for the energy density and pressure given in Eqs. (186) and (187):

$$\rho = -\alpha^2 F^2 + \frac{1}{4}F + \frac{\mu^2}{F} - \frac{\beta^2}{F^2},$$
(262)

$$p = -\frac{5\alpha^2}{3}F^2 + \frac{1}{12}F - \frac{7\mu^2}{3}\frac{1}{F} + \frac{11\beta^2}{3}\frac{1}{F^2}.$$
(263)



Fig. 12. Qualitative plot of the effective potential for $D_V > 0$, $D_{V'} = 0$. The lower dotted line corresponds to $\epsilon = 1$.

As we saw in Section 4.5.3, for any Lagrangian that is a polynomial in F,

$$\mathscr{H}=\mathscr{H}_0\,a^{-2}.$$

As discussed in [328], the combined system of equations of the FLRW metric and the magnetic field described by General Relativity and NLED, are such that the negative energy density contributions coming from \mathcal{L}_1 and \mathcal{L}_4 never overcome the positive terms arising from \mathcal{L}_2 and \mathcal{L}_3 . Before reaching undesirable negative energy density values, the universe bounces (for very large values of the field) and bounces back (in the other extreme, that is, for very small values) to precisely avoid this difficulty. These events occur at the values $\rho_B = \rho_{TA} = 0$, which follow from Friedmann's equation in the case $\epsilon = 0$. Notice that this is not an extra condition imposed by hand but a direct consequence of the dynamics described by \mathcal{L}_T .

Let us now turn to the generic conditions needed for the universe to have a bounce and a phase of accelerated expansion. From Einstein's equations, the acceleration of the universe is related to its matter content by

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p).$$
(264)

In order to have an accelerated universe, matter must satisfy the constraint ($\rho + 3p$) < 0, which translates into

$$\mathcal{L}_F > \frac{\mathcal{L}}{4\mathscr{H}^2}.$$
(265)

It follows that any nonlinear electromagnetic theory that satisfies this inequality yields accelerated expansion. In the present model, the terms \mathcal{L}_2 and \mathcal{L}_4 produce negative acceleration and \mathcal{L}_1 and \mathcal{L}_3 yield inflationary regimes ($\ddot{a} > 0$). Raychaudhuri's equation imposes further restrictions on a(t) at a bounce. Indeed, the existence of a minimum (or a maximum) for the scale factor implies that at the bounce point *B* the inequality (ρ_B+3p_B) < 0 (or, respectively, (ρ_B+3p_B) > 0) must be satisfied. Note that, as already mentioned, at any extremum (maximum or minimum) of the scale factor the energy density is zero. Four distinct periods can be identified according to the dominance of each term of the energy density. The early regime (driven by the F^2 term); the radiation era (where the equation of state $p = 1/3\rho$ controls the expansion); the third accelerated evolution (where the 1/F term is the most important one) and finally the last era where the $1/F^2$ dominates and in which the expansion stops, the universe bounces back and starts to collapse. The bounce (for an Euclidean section) was discussed in Section 4.5.3. The standard, Maxwellian term dominates in the intermediate regime. Due to the dependence on a^{-2} of the field, this phase is defined by $\mathcal{H}^2 \gg \mathcal{H}^4$ yielding the approximation

$$\rho \approx \frac{\mathscr{H}^2}{2}$$

$$p \approx \frac{\mathscr{H}^2}{5}.$$
(266)

When the universe becomes larger, negative powers of *F* dominate and the energy density becomes typical of an accelerated universe, that is:

$$\rho \approx \frac{1}{2} \frac{\mu^{*}}{\mathscr{H}^{2}}$$

$$p \approx -\frac{7}{6} \frac{\mu^{*}}{\mathscr{H}^{2}}.$$
(267)

In the regime between the radiation and the acceleration eras, the energy content is described by

$$\rho = \frac{\mathscr{H}^2}{2} + \frac{\mu^2}{2} \frac{1}{\mathscr{H}^2},$$

or, in terms of the scale factor,

$$\rho = \frac{\mathscr{H}_0^2}{2} \frac{1}{a^4} + \frac{\mu^2}{2\mathscr{H}_0^2} a^4.$$
(268)

For small *a* it is the ordinary radiation term that dominates. The 1/F term takes over only after $a = \sqrt{\mathcal{H}_0}/\mu$, and grows without bound afterwards. Using this matter density in Eq. (264) gives

$$3\frac{\ddot{a}}{a} + \frac{\mathscr{H}_0^2}{2} \frac{1}{a^4} - \frac{3}{2}\frac{\mu^8}{\mathscr{H}_0^2} a^4 = 0.$$

To get a regime of accelerated expansion, we must have

$$\frac{\mathscr{H}_0^2}{a^4} - 3\frac{\mu^8}{H_0^2} a^4 < 0,$$

which implies that the universe will accelerate for $a > a_c$, with

$$a_c = \left(\frac{H_0^4}{3\mu^8}\right)^{1/8}.$$

For very large values of the scale factor, the energy density can be approximated by

$$\rho \approx \frac{\mu^2}{F} - \frac{\beta^2}{F^2} \tag{269}$$

and the model goes from an accelerated regime to a phase in which the acceleration is negative. When the field attains the value $F_{TA} = 16\alpha^2 \mu^2$ the universe stops expanding and turns to a collapsing phase. The scale factor attains its maximum value

$$a_{\max}^4 \approx \frac{\mathscr{H}_0^2}{8\alpha^2\mu^2}.$$

Analytic forms for the scale factor in each regime can be found in [328].

10.2.3. Cyclic universes in loop quantum gravity

There are realizations of cyclic models in the effective equations for loop quantum gravity (some features of which have been presented in Section 9.2). As discussed in Section 9.2, the Klein–Gordon equation for a scalar field under the influence of a potential, the Friedmann and Raychaudhuri's equations in the semiclassical regime are modified due to quantum gravity effects (see Eqs. (254)–(256)). It was shown in [258] that positively curved universes sourced by a massless scalar field can undergo repeated expansion and contraction due to the modifications described above. This was achieved by rewriting Eqs. (254)–(256) in the form of the classical FLRW model with the addition of matter described by an effective equation of state, given by

$$\omega \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2DV} \left(1 - \frac{1}{6}\frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\right).$$

A violation of the null energy condition, leading to a bounce, is accomplished when $\omega < -1$, which amounts to d ln $D/d \ln a > 6$, or $a < 0.914a_*$ [258], with

$$D(q) = \left(\frac{8}{77}\right)^{\circ} q^{3/2} \{7[(q+1)^{11/4} - |q-1|^{11/4}] - 11q[(q+1)^{7/4}] - |q-1|^{7/4} \operatorname{sgn}(q-1)\}^{6},$$

with $q = a^2/a_*^2$ and $a_*^2 = \gamma \ell_{pj}^2 j/3$, where $\gamma \approx 0.13$, and *j* is a quantization parameter, which takes half-integer values. When V = 0, ω is independent of the kinetic energy of the field, and an oscillatory behaviour follows. The addition of a potential leads to the interruption of the cycles as soon as the potential dominates the motion (in analogy to what was discussed in Section 10.1 for the cosmological constant), and a period of inflation may follow [258]. This analysis was later extended to the case of spatially flat universes, with both negative an positive potentials [297,378].

Yet another realisation of a cyclic universe in this scenario is the so-called *emergent universe from a loop* [298]. As mentioned in Section 8.3, the Einstein universe is unstable, so perturbations drive the universe away from this state. This situation partially changes when loop quantum gravity corrections are considered. Using a phase-space analysis, it was shown in [298] that a new static solution appears in the semiclassical regime ($a < a_*$) for positive potentials (for V < 0 this is the only solution). This new solution (called loop static, LS) is stable, and the universe oscillates around it, for $V < V_*$, with $V_* = 39/(136\pi l_{Pl}^2 a_*^2)$, while for $V > V_*$, the equilibrium point corresponding to LS merges with that of the Einstein universe. So in the model proposed in [298], the universe is initially at, or in the neighbourhood of the static point LS, with ϕ in the plateau region of the potential with $\dot{\phi}_i > 0$. After undergoing a series of non-singular oscillations in a (possibly) past-eternal phase, while the field evolves monotonically along the potential, the cycles are eventually broken as the magnitude of the potential increases, and the universe enters an inflationary epoch. For this model to work, the potential must be such that $dV/d\phi \rightarrow 0$ for $\phi \rightarrow -\infty$, and increase monotonically to exit the cycles.⁸² An example of a suitable potential is given by

$$V = \alpha \left[\exp(\beta \phi / \sqrt{3}) - 1 \right]^2,$$

where α and β are parameters that may be constrained by the CMB spectrum. As in the case of the classical emergent universe discussed in Section 8.3, there are some fine-tuning issues: the scalar field must start in the asymptotically low-energy region of *V*.

10.2.4. The cyclic universe based on the ekpyrotic universe

The starting point of the ekpyrotic scenario [241] is five-dimensional heterotic M-theory [211], where the fifth dimension terminates at two boundary Z_2 branes, one of which is identified with the visible universe. There are two different versions of the ekpyrotic scenario, the old [240], where there is a bulk brane between the boundary branes and the new [242], where only boundary branes are present [348]. The initial state in both cases is supposed to be the vacuum state, where branes are flat, parallel and empty. The branes are drawn together by the action of an attractive potential, and collide inelastically over cosmological times. Part of the kinetic energy is transferred to the branes and used to create matter and radiation. After the collision, the universe enters a "standard" big bang phase, until dark energy domination at the end of the matter era, which causes an accelerated expansion, diluting the content of the universe. The whole process can be described by a 4-d effective theory, with the action (in the Einstein frame) given by

$$S_E = \int d^4x \sqrt{-g} \left(\frac{1}{2}R - \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right),$$

plus higher-order corrections, where the conveniently-tailored potential $V(\varphi)$ is responsible for the main features of the model. The potential is slightly positive for $\varphi > 0$, and goes to zero as $\varphi \to -\infty$. For $\varphi < 0$, the potential has a minimum and is very steep and negative. The minimum corresponds to the close approach of the branes, which happens at such short distances that quantum gravity effects are relevant. The field φ moves rapidly through the minimum, and the branes collide as $\varphi \to -\infty$. Both the old and the new model were shown to have problems due to excessive fine-tuning [226], so a cyclic version was introduced [386].

In the cyclic ekpyrotic model [387], it is assumed that the interbrane potential is the same before and after collision (instead of being zero, as in the non-cyclic model). After the branes bounce and fly apart, the interbrane potential ultimately causes them to draw together and collide again. To ensure cyclic behavior the potential must vary from negative to positive values [386]. The model may be adjusted in such a way that, at distances corresponding to the present-day separation between branes, the inter-brane potential energy density is positive and corresponds to the currently observed dark energy, providing roughly 70% of the critical density today. The cosmic acceleration restores the Universe to a nearly vacuous state and as brane separation decreases, the interbrane potential becomes negative. As the branes approach one another, the scale factor of the Universe, in the conventional Einstein description, changes from expansion to contraction. When the branes collide and bounce, matter and radiation are produced and there is a second reversal transforming contraction to expansion so a new cycle can begin [386]. Fig. 13 shows a plot of several forms of the potential that allow for a cyclic universe in this scenario [243]. A qualitative description of the model can be given in terms of this figure as follows. Currently, the field is in region (a), at the point indicated with a dark circle, where the potential is flat and drives cosmic acceleration. Eventually, the field rolls towards negative values of V (region b), where cosmic expansion stops and the universe (being nearly vacuous as a consequence of the acceleration phase) enters a phase of slow contraction, where the spectrum of density perturbations is generated from quantum fluctuations in φ . In region (c) the kinetic energy of φ dominates the energy density. At the bounce, part of this kinetic energy is converted into matter and radiation, while the perturbations in φ are imprinted as density fluctuations in the matter/radiation fluid. Meanwhile the field quickly returns back to (a) where it comes to a stop, and the universe enters the radiation-dominated era, so commencing the next cycle. As recognized by its authors, the model presents two weak points (as is the case with many cyclic models): passage through the would-be singular point, and the propagation of perturbations.⁸³ It is difficult to achieve the bounce without passing from the semi-classical regime to the high-energy fully quantum regime, where our use of the effective 4-dimensional theory breaks down. The problem is that the kinetic energy and the Hubble rate typically reach Planckian scale as the branes approach. In fact, in the semi-classical regime where loop corrections can be applied, brane collision may be prevented.⁸⁴

Recently, a "new ekpyrotic cosmology" was presented in [87], where a NEC-violating ghost condensate was merged with an ekpyrotic phase to generate a non-singular bouncing cosmology. The authors claim to obtain a pre-bounce scale-invariant spectrum using the mechanism of entropy perturbation generation [251]. This is accomplished by having two ekpyrotic scalar fields rolling down their respective negative exponential potentials, and having its own higher-derivative kinetic function. Notice however that the results of this model have been challenged in [227].

⁸² Other constraints are imposed by successful reheating.

⁸³ This second problem will be discussed in Section 11.

⁸⁴ Some other problems of the model were discussed by Linde [262].



Fig. 13. Plot of several allowed forms for $V(\varphi)$.

10.2.5. Oscillatory universe from the steady state model

The Steady State model [68] was proposed as alternative to the Big Bang model, and has fallen into disfavor because the observations of the CMB. However, its authors have advanced a new scenario, called the quasi-steady state model (QSSC, see [214,362,303]). In this model, the singularity is avoided by the action of a scalar field C(x), which creates matter in compliance with the Weyl postulate and the cosmological principle, and has negative energy and stresses. Cyclic solutions in the QSSC can be expected from physical grounds as follows [362]. To create a particle, C(x) must have energy-momentum equal or larger than that of the particle. When *C* is above the threshold, it creates particles and fuels spacetime expansion (since it has negative stresses). To this overall expansion an oscillation is superimposed. The creation of particles and the expansion set *C* below the threshold, slowing down the number of created particles, and the expansion. Here, the cosmological constant takes control and causes contraction. The contraction raises the background level of the *C* field, and the cycle starts again. As shown in [362], a solution to the EOM of this theory in the FLRW setting that oscillates in this way is given by

$$a(t) = e^{t/P} \left(1 + \eta \cos \theta(t) \right),$$

with $\theta(t) \approx 2\pi t/Q$, where *P* is the long term "steady state" time scale of expansion, *Q* is the period of a single oscillation (with $P \gg Q$), and η is a parameter.

10.2.6. Other models

Due to the recently discovered dark energy component of the universe, several forms for the dependence of the EOS parameter with redshift have been analyzed [201]. In fact, some data suggest that $\omega(z)$ evolved from a value larger that -1 to a value smaller that -1 at some recent redshift. One of the models that describes this crossing is the quintom model [160], where ω is parameterized as⁸⁵

$$\omega(\ln a) = \omega_0 + \omega_1 \cos[A \ln(a/a_c)],$$

(270)

with ω_0 , ω_1 , A, and a_c to be fitted by observations.⁸⁶ It was shown in [161] that for a certain choice of the parameters, a universe filled with quintom matter (that is, matter with ω given by Eq. (270)) plus radiation and normal matter expands and contracts cyclically, yielding an inflationary period at the beginning of each cycle, and an acceleration period at the end.⁸⁷

Perhaps it is convenient at this point to remember that a closed universe has not been discarded by observation yet (and in fact, cannot be discarded with certainty due to the errors inherent to any experiment), though theoretical prejudice and observation tend to favor $\Omega = 1$. As we saw in Section 5, a nonzero bulk viscosity ζ modifies the fluid pressure according to

$$p=p_0-3\zeta H,$$

where p_0 is the equilibrium pressure. The asymmetry in the pressure depending on the sign of *H* causes the increment in energy and entropy, leading to ever-increasing cycles. It was shown in [231] that a similar asymmetry can be caused by scalar fields in a pure non-dissipative setting. Starting from a FLRW setting plus a scalar field under the influence of a potential

⁸⁵ Constraints on this form of dark energy were studied in [266].

⁸⁶ Similar ideas were studied in [29,308].

⁸⁷ Cosmological perturbations of the quintom model were studied in [421].

which displays a minimum, an asymmetry in the pressure, given by $p \approx -\rho$ for H > 0, and $p \approx \rho$ for H < 0 is generated by the oscillations of the field around the minimum [231]. By imposing appropriate conditions to force a bounce $(a \rightarrow a, \dot{a} \rightarrow -\dot{a}, \phi \rightarrow \phi, \dot{\phi} \rightarrow \dot{\phi})$, it was shown that there is an in increment in the maximum radius of expansion of the universe in each cycle, due to conversion of work, done during expansion, into expansion energy. The flatness problem is gradually ameliorated in this model, since the universe becomes considerably more long-lived and more flat after each expansion.

To close this section, other models of a cyclic evolution for the universe are listed next⁸⁸:

- String theory-inspired cyclic universes, starting from the property that there exists a minimal length, ℓ_{Pl} . See [198].
- A classical spinor field under the influence of a quartic potential in a FLRW background was discussed in [22]. It was
- shown that $V = \lambda_{\psi} + m\psi\bar{\psi} \lambda(\bar{\psi}\psi)^2$ gives rise to oscillations in the scale factor, for certain choices of the parameters.⁸⁹ • A cyclic scenario that takes into account matter and radiation evolution if the proton has a finite lifetime was studied in [133].

10.3. Issues of the cyclic models

Cyclic universes are not free of problems. As was put forward in [337], during a matter-dominated cycle, black holes with masses ranging from stellar to galactic will form. During the contracting phase they will coalesce into a "monster black hole" with mass equal to the mass of the universe. Its entropy can be estimated by

$$S = rac{1}{2}A = 2\pi R^2 = 8\pi M^2 \gtrsim 10^{124},$$

where the mass within one Hubble volume ($\approx 10^{23}M_{\odot}$) was used. However, the entropy of the radiation in the present Hubble volume is $\approx 10^{87}$, in such a way that black hole formation in a previous cycle would lead to a huge excess of entropy generation. In this scenario, the excess must have somehow been eliminated by the bounce. But there are some ways out of this problem. Sikkema and Israel [375] have suggested that the inner horizon of the monster Kerr black hole absorbs strongly blue-shifted gravitational radiation emitted during the last moments of the collapse. This radiation increases the mass of the core of the black hole by a huge amount, rapidly reaching Planckian values, and correspondingly greatly reduces its specific entropy. If quantum effects produce a bounce, this process would allow the expansion to begin from a state of relatively low disorder Durrer and Laukenmann [138] have noted that the entropy in the radiation we observe today is actually due to the *previous* matter cycle, which may have had a shorter duration than the current cycle, leading to less clumping and consequently less entropy production.⁹⁰

Another issue of cyclic models was raised in [25], where the evolution of a cosmic string network was considered in a bouncing universe. It was shown that the string network displays asymmetric behaviour between the contraction and expansion epochs. In particular, while during expansion a cosmic string network will quickly evolve towards a linear scaling regime, in a phase of collapse it would asymptotically behave like a radiation fluid. A cosmic string network will add a significant contribution, in the form of radiation, to the energy (and hence also entropy) budget of a contracting universe, which will become ever more important as the contraction proceeds. Hence it establishes the need for a suitable entropy dilution mechanism. This process will also operate, *mutatis mutandis*, for other stable topological defects. Conversely, if direct evidence is found for the presence of topological defects (with a given energy scale) in the early universe, their existence alone will impose constraints on the existence and characteristics of any previous phases.

11. Perturbations in bouncing universes

As discussed in the introduction, inflation can solve many of the shortcomings of the SCM, but it also has problems of its own. Bouncing models may provide an alternative (or perhaps a complement) to standard inflation, since in principle the problems of the SCM come from a "shortage of time" for things to happen early after the big bang [179]. The arguments in Section 1 show that an accelerated contraction has the necessary features to solve the problems of the SCM [173]. Let us recall that if in the contracting phase the Hubble radius decreases faster than the physical wavelength corresponding to fixed comoving scales, quantum fluctuations on microscopic scales can be stretched to scales which are cosmological at the present time, exactly as it happens in inflationary models (see for instance [179]). Fig. 14 shows a sketch of the structure of a space–time in which standard inflation starts at t_i and ends at t_R . During inflation, the Hubble radius $H^{-1}(t)$ is constant, and it grows linearly afterwards, while the physical length corresponding to a fixed co-moving scale increases exponentially during the period of inflation, and then grows less fast than $H^{-1}(t)$. The figure shows that for a given k, the fluctuation can be (causally) produced well inside the Hubble radius, "leave" $H^{-1}(t)$, and "re-enter" in an appropriate way to describe the structures we observe today.

⁸⁸ See also [309].

⁸⁹ See also [422].

⁹⁰ Gravitational perturbations were also studied in [138].



Fig. 14. Behavior of the comoving scale k and of the Hubble radius H^{-1} as a function of time in inflation. Taken from [79].

Fig. 15 shows a universe that undergoes a contracting phase, a bounce, and then enters an expanding epoch, assumed to be that of the SCM. In this case, the Hubble radius decreases relative to a fixed comoving scale during the contracting phase, and increases faster in the expanding phase. Fluctuations of cosmological interest today are generated sub-Hubble but propagate outside the Hubble radius for a long time interval. There is however, one main difference with respect to the standard inflationary scenario. In the latter the curvature scale $R \propto H^2$ is (almost) constant [270], while in the former, it grows until it reaches a maximum and then decreases.⁹¹ This difference may lead to observational consequences,⁹² particularly regarding the generation of a primordial spectrum of inhomogeneities through parametric amplification of quantum fluctuations of background fields in their vacuum state [293]. These, when decomposed in Fourier modes, satisfy a canonical Schrodingerlike equation, whose effective potential is determined by the so-called "pump field", which depends in its turn on the background geometry. There are then two properties of the background in a bouncing universe that can affect the final form of the perturbation spectra [177]: (1) the growth of the curvature scale, and (2) the fields which, together with the gravitational field, determine the background. Property (1) has two important consequences. The first, is that bouncing scenarios may lead to "blue" (i.e. growing with frequency) metric perturbation spectra, instead of being flat, or decreasing ("red"), as in standard inflation. A growing spectrum leads to the formation of relic backgrounds whose amplitude is higher at higher frequency, hence more easily detectable. A typical example is that of gravitational waves in SPPB [177] (see Eq. (279)). The second is that the growth of the curvature may also force the comoving amplitude of perturbations to grow (instead of being frozen) outside the horizon (see [84] for this effect in the SPBB).⁹³

Regarding Property 2, one of the interesting consequences is amplification of the fluctuations of the EM field, due for instance to non-minimal coupling with a scalar field (such as the dilaton, or the scalar field in WIST, see Section 11.4). A relic background of scalar particles is also generated, which may be related to dark matter [174].

There is yet another salient feature of perturbations in a bouncing universe. Since in the far past of this type of models the universe is assumed to be almost flat, one can impose vacuum initial conditions for perturbations based on simple quantum field theory in flat space [344], instead of having to set initial conditions in a high-curvature regime.

It must be remarked that solving for perturbations in bouncing models is in principle a nontrivial task, since there are potential ambiguities that may arise at the bounce, not present in standard inflation.⁹⁴ Two views can be taken to tackle the

⁹¹ This assertion is valid in models in which quantum effects intervene in such a way that $R_{max} \propto \lambda_{min}^{-2}$, which is the case of loop quantum gravity for instance, where $\lambda_{\min} \propto \ell_{Pl}$. For the models in which *H* reaches a null value, H^2 can be replaced by $\ell_c = \sqrt{a^3/a''}$, see Eq. (355).

⁹² See [177] for a qualitative discussion of these consequences in the case of string pre-big-bang cosmology.

⁹³ Consequently, special attention must be taken in the application of linear perturbation theory, see [84].

⁹⁴ For instance. at the bounce the comoving Hubble scale diverges. Hence all scales are inside the Hubble scale, at least for an instant. However, there are some issues common to both scenarios, such as the transplanckian problem (see for instance [78]).



Fig. 15. Behavior of the comoving scale k and of the Hubble radius H^{-1} as a function of time in a bouncing universe. Taken from [79].

study of perturbations in such a scenario. The first one is to devise first a detailed model of the bounce, and then study the properties of post-bounce perturbations. The problem in this case is that total control of the high-energy physics involved in the bounce is needed, which is not always achieved. It may also happen that the bouncing solution under scrutiny is quite artificial from the physical point of view, as for instance if it is not embedded in any fundamental theory. But in any case some lessons may be extracted from the examples, as we shall see in Section 11.1.

A second attitude is to make some simplifying assumptions and try to work out predictions that are independent of the UV physics that most surely governs the bounce. This possibility has led to a great debate [272]. In particular, in order to avoid the specification of the details near the high-curvature regime, matching conditions are used, leading to ambiguities. The dependence of the post-bounce spectrum on matching conditions has been addressed by many authors, as will be discussed in Section 11.3.

At this point, it is perhaps necessary to say that there are at least two alternative procedures to deal with gravitational perturbations in a relativistic setting. Since Lifshitz's original paper [259], it has been a common practice to start the examination of the theory of perturbations of General Relativity by considering variations of non-observable quantities, such as $\delta g_{\mu\nu}$. The main drawback of this procedure is that it mixes true perturbations and arbitrary (infinitesimal) coordinate transformations, which are unphysical. As shown in [28,223,74,293], this problem can be solved by adopting gauge-independent combinations of the perturbed quantities expressed in terms of the metric tensor and its derivatives. The dynamics of these gauge-independent variables is then provided by the EE.

A second method exists, based on the quasi-Maxwellian (QM) formulation of Einstein's equations. The advantage of this method is that it is gauge-independent from the start, thus dealing only with observable quantities [204,329,318–320,216, 149]. We shall briefly review both methods in Section 11.5, including a summary of the relation between them.

In the next sections we shall discuss examples of the two approaches. From an observational point of view, the crucial question is whether bouncing models can furnish a nearly-scale invariant spectrum of adiabatic scalar perturbations after the bounce, as demanded by the measurements of the WMAP [380], Sloan survey [391], and 2df [338]. It is also of interest to see if bouncing solutions lead to observable consequences that are markedly different from those of inflation (see Section 12).

11.1. Regular models

In the previous chapters, we have seen that it is possible to generate bouncing models in a wide choice of scenarios, essentially by any of the mechanisms presented in Section 1.2. Obviously, the outcome is very dependent on the choice, but specific models can be sometimes useful in the hope of extracting tendencies of a more general behaviour. In this sense, scalar, vector, and tensor perturbations have been studied in many exact backgrounds displaying a bounce. An incomplete list includes the following:

• General relativity with radiation and a free scalar field having negative energy [340].

- Two scalar fields [200,9,134].
- A 5d Randall-Sundrum model with radiation, in which the extra dimension is timelike [40].
- Two perfect fluids [163].
- A nonlinear EM Lagrangian [326].
- A scalar field with higher-order corrections from string theory, with an exponential potential (this case covers the SPBB and the first version of the Ekpyrotic universe) [99,400].
- A non-canonical scalar field, with Lagrangian $\mathcal{L} = p(X, \phi)$, where $X = 1/2g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$ [4,407].
- Bounce due to quantum cosmological effects using Bohmian solutions of the canonical Wheeler-de Witt equation [343].
- Non-local dilaton potential stemming from string theory [178].

We shall present next a short discussion of scalar, tensor, and vector perturbations in some of these scenarios.

11.1.1. Scalar perturbations

The evolution of scalar perturbations through a bounce has been a subject of intense debate (see references in [72]). A consensus for the case of a two-component bouncing model in GR seems to have been reached after [72]. This model is described by a flat FLRW metric, and one of the components has negative energy density (to produce the bounce) and is important only near the bounce. The components interact only gravitationally, and the component that dominates away from the bounce has an intrinsic isocurvature mode, in order to describe scalar fields or perfect fluids. The result obtained in [72] is that the spectrum of the growing mode of the Bardeen potential in the pre-bounce is transferred to a decaying mode in the post-bounce.^{95,96}

Since the phenomenology associated with the decaying mode in the pre-bounce phase is known to differ from observation [12], we may ask what can be done to lift the negative result of [72]. One possibility is to allow the fluids to interact. Another one is to incorporate in the background solution the decay of the normal component to radiation.⁹⁷ Yet another possibility is to consider higher-order corrections. This has been done in several string-inspired models,⁹⁸ in the gravi-dilaton regime by exploring regular backgrounds (such as those presented in Section 3.3.1), as in [98,399,400,100]. The results presented in these articles show that although it may be possible to generate a nearly scale-invariant spectrum in the pre-bounce phase, it corresponds to the decaying mode in the expanding phase.⁹⁹ An exception is the model presented in Section 11.1.1. Another exception may be the ekpyrotic model, where there are results indicating that a scale-invariant spectrum may be obtained in the post-bounce phase [394].¹⁰⁰

Another set of models comes from the quantum evolution of the universe. As discussed in Section 9.1, bouncing solutions are possible (without the need of a "phantom" field) in the context of the WdW equation, when the Bohm-de Broglie interpretation is used in the mini-superspace approach. A feature of this scenario is that a full quantum treatment of both background and perturbations is possible [341,342]. The model analyzed in [343] is GR plus a perfect fluid, in which the scalar perturbations can be described in terms of a single degree of freedom, related to the Bardeen potential Φ (see Appendix). The Bohmian quantum trajectory for the scale factor is given by

$$a(T) = a_0 \left[a + \left(\frac{T}{T_0}\right)^2 \right]^{\frac{1}{3(1-\omega)}},$$
(271)

with $p = \omega \rho$. The normal modes of the scalar perturbation satisfy the equation

$$v_k'' + \left(k^2 - \frac{a''}{a}\right)v_k = 0,$$
(272)

where a prime means derivative w.r.t. conformal time. Following the usual procedure of expanding the modes for large (negative and positive) values of *T*, matching the expansions, and then transforming to the Bardeen potential, the power spectrum defined by

$$P_{\Phi} = \frac{2k^3}{\pi^2} |\Phi^2| \propto k^{n_s - 1},\tag{273}$$

yields for the post-bounce phase [343]

$$n_{\rm s} = 1 + \frac{12\omega}{1+3\omega}.$$
 (274)

⁹⁵ These result is supported by the references cited in [72] and also by the results in [163].

⁹⁶ Notice that mode-mixing is possible with $\epsilon = 1$, as for instance in [217].

⁹⁷ See Section 3.2.9 and [364] for an exact solution that has this feature.

⁹⁸ The string pre-big-bang model without corrections furnishes a highly blue-tilted spectrum $n_s = 4$ of scalar perturbations [84].

⁹⁹ The SPBB model may yield the right spectrum when axion fluctuations are considered [154].

¹⁰⁰ See Ref.[8] for another model in which the growing mode in the contracting phase goes over into the dominant mode in the post-bounce phase.

An analogous calculation for the tensor modes gives

$$n_T = \frac{12\omega}{1+3\omega}.$$
(275)

Notice that a scale-invariant spectrum follows both for the scalar and the tensor perturbations for the case of dust ($\omega = 0$), which is the fluid supposed to dominate the evolution at the time of the matching of the solutions (not necessarily the same governing at the time of the bounce) [343]. An important lesson that follows from this example and the one presented in [163] (see Section 11.1.3) is that the spectral index is quite insensitive to the details of the bounce, being determined mostly by the dominant component. The example also shows that the bounce is important in the mixing of the modes, which is relevant for the amplitude of the modes in the post-bounce phase.

11.1.2. Vector perturbations in a contracting background

It is a well-known result of perturbation theory that vector perturbations (VPs) only exhibit decreasing solutions in the context of an expanding Universe (see for instance [293]).¹⁰¹ However, as shown in [39], VPs can increase in a contracting flat background, with a perfect fluid as source. Hence, they might provide a signature of a bounce. As shown in the Appendix, the relevant equations are $S_k^i = C_k^i/a^2$, where *C* is a constant, and

$$V_k^i \propto \frac{k^2 C_k^i}{a^{1-3\omega}}.$$
(276)

Note that V_k^i increases for $\omega = 0$, and stays constant for radiation, but S_k^i always increases for decreasing *a*. As argued in [39], VPs cannot be neglected in the SPBB scenario, in such a way that near a bounce, metric perturbations may become too large for the use of linear theory (depending on the value of the C_k^i).¹⁰² Related results were presented in [186], where it was also shown that the growing vector mode matches with a decaying mode after the curvature bounce, in the context of a low-energy flat gravi-dilaton model.¹⁰³

Since many bouncing models are generated by a scalar field, a relevant question is whether VPs are important in this type of scenarios. One important point is that VPs are not supported by a scalar field at first order. At second order, the scalar, vector, and tensor modes couple, and VPs can be generated by scalar–scalar mode couplings [287]. Considering exponential potentials and power-law solutions, the ratio of the amplitudes of second order vector perturbations in contracting and expanding phases was studied in [287]. The relative magnitudes of the second order vector perturbations in the two phases depend on the scaling solutions chosen, but at least in one of the examples studied (dust-like collapse, [162]), the observable differences between the collapsing models and the inflationary scenario could be large, assuming that the transition between the two phases does not significantly alter the ratio.

11.1.3. Tensor perturbations

The spectrum of gravitational waves can be a very powerful tool to discriminate between different models of the universe, since gravitational waves decouple very early from matter and travel undisturbed, as opposed to EM waves. In particular, in the context of the SPBB scenario, amplification of tensor perturbations is greatly enhanced w.r.t. the standard inflationary scenario for large comoving wavenumber *k* [171]. This result was confirmed in [84], with a gravi-dilaton background solution of the EOM

$$G^{\nu}_{\mu} = \frac{1}{2} \left(\partial_{\mu} \varphi \partial^{\nu} \varphi - \frac{1}{2} \delta^{\nu}_{\mu} \partial_{\alpha} \varphi \right),$$

$$\Box \varphi = 0,$$
(277)
(278)

 $\Box \varphi = 0,$

given by

$$a(\eta) = (-\eta)^{1/2}, \qquad \varphi(\eta) = \frac{-3 - \sqrt{3}}{1 + \sqrt{3}} \ln(-\eta) + \text{const.},$$

the typical amplitude for the normalized vacuum tensor fluctuations outside of the horizon over a scale k^{-1} is given by [84]

$$|\delta_{h_k}(\eta)| \approx \left(\frac{H_1}{M_{Pl}}\right) (k\eta_1)^{3/2} \ln |k\eta|, \tag{279}$$

where $H_1 \approx (a_1\eta_1)^{-1}$ is the final contraction scale,¹⁰⁴ while the result in standard inflationary expansion does not have the ln dependence (see for instance [192]). The possible influence of the nonperturbative phase, where the curvature and the

¹⁰¹ Another interesting result is that the simplest models of inflation do not produce VPs, see for instance [265].

¹⁰² Quantum corrections to the evolution of vector modes were studied in the context of loop quantum gravity in [67].

¹⁰³ This is not necessarily so in multidimensional cosmological models, also analyzed in [186].

¹⁰⁴ Scalar perturbations of this model were also investigated in [84], and present amplitudes and spectra similar to tensor perturbations.

dilaton are very large, was studied by imposing a bouncing solution in [169], and by taking into account higher-derivative α' and quantum corrections (see Section 3.3.1) [176,97]. The results in these papers show that low frequency modes, crossing the horizon in the low-curvature regime, are unaffected by higher-order corrections, and also that the shape of the spectrum of the relic graviton background, obtained in the context of the pre-Big Bang scenario, is strongly model-dependent.

This analysis was continued in [427], where cosmological perturbations in the low-energy string effective action with a dilaton coupling $F(\phi)$ were studied, with the addition of a Gauss–Bonnet term, a kinetic term of the type $(\nabla \phi)^4$, and a potential $V(\phi)$. Scale-invariant spectra in the string frame and a suppressed tensor-to-scalar ratio were obtained by imposing slow-roll inflation in the Einstein frame. The results show that it is practically impossible to obtain these conditions without the second-order corrections given by Eq. (154), both with and without the Gauss–Bonnet term.

Analytic and numerical results for the tensor post-bounce spectrum have been obtained for a two-component model defined by $p_{\pm} = \omega_{\pm} \rho_{\pm}$ [163]. The flat background is given by

$$a(\tau) = a_0 \left(1 + \frac{\tau^2}{\tau_0^2}\right)^{\alpha},$$

with

$$\begin{aligned} &d\tau = \frac{dt}{a^{\beta}}, \qquad \beta = \frac{3}{2}(2\omega_{+} - \omega_{-} + 1), \\ &\alpha = \frac{1}{3(\omega_{-}\omega_{+})}, \qquad a_{0} = \left(\frac{\gamma_{-}}{\gamma_{+}}\right)^{\alpha}, \qquad \tau_{0}^{2} = \frac{4\alpha^{2}}{\ell_{\mathrm{Pl}}^{2}}\frac{\gamma_{-}}{\gamma_{+}^{2}} \end{aligned}$$

 γ_+ and γ_- are constants, with $\gamma_- < 0$, to produce the bounce. The tensor spectrum, assuming that $-1/3 < \omega_+ < 1$, and that the potential that arises from Eq. (355) has only one extremum at $\tau = 0$, is given by [163] $P_h \propto \tilde{k}^{n_T}$, where

$$n_T=\frac{12\omega_+}{1+3\omega_+}.$$

Note that the spectral index does not depend on the EOS parameter of the "exotic" fluid (contrary to the case of the spectral index for the scalar perturbations). This was to be expected since large wavelengths are comparable to the curvature scale of the background at a time when the universe is still far from the bounce, so the behaviour obtained in this case can be taken as generic, i.e. independent of the details of the bounce.

Yet another example of the calculation of a tensor spectrum in a bouncing model was presented in Section 11.1.1, based on the quantum evolution (using the Bohmian quantum trajectory) of a universe described by GR plus a perfect fluid. The result is (see the comments after Eq. (275))

$$n_T = \frac{12\omega}{1+3\omega}.$$
(280)

In fact, the tensor-to-scalar ratio in this model was estimated as $T/S \approx 5.2 \times 10^{-3}$, and the characteristic bounce length-scale $L_0 \approx 1500\ell_{Pl}$, (assuming that $n_s \leq 1.01$) which is a value in the range in which quantum effects are expected to be relevant, while at the same time the Wheeler-de Witt equation is valid (without corrections from stringy/loop effects).

11.2. Scalar perturbations in exact models using the quasi-Maxwellian framework

As mentioned in the introduction of this chapter, perturbations can also be studied using the quasi-Maxwellian (quasi-Maxwellian) method. In this section we apply it to two exact bouncing solutions. The first one is generated by non-minimal coupling of the electromagnetic field with gravity (see Section 4.4). As discussed in the Appendix, in quasi-Maxwellian formalism scalar perturbations are completely described by the variables *E* and Σ , which obey the Eqs. (376)–(378):

$$\dot{E} = -\frac{1+\lambda}{2}\rho\Sigma - \frac{1}{3}\theta E,$$

$$\dot{\Sigma} = \left[\frac{6\lambda}{1+\lambda}\left(\epsilon + \frac{k^2}{3}\right)\frac{1}{a^2\rho} - 1\right]E,$$

. .

with $p = \lambda \rho$, and k is the wave number (the subindex k in E and Σ has been omited). Combining these, we obtain the equation for the time evolution of the electric part of the perturbed Weyl tensor:

$$\ddot{E} + \dot{E} \left(\frac{4}{3} + \lambda\right) \theta + EX = 0, \tag{281}$$

where X is a function of the background variables given by

$$X \equiv \lambda \frac{3\epsilon + k^2}{a^2} - \left(\lambda + \frac{2}{3}\right)\rho + \frac{2 + 3\lambda}{9}\theta^2.$$

Defining a new function g(t) by $g = E a^{-\sigma}$, where $\sigma \equiv -(4 + 3\lambda)/2$, we obtain from Eq. (281)

$$\ddot{g} + \chi(t) g = 0, \tag{282}$$

where¹⁰⁵

$$\chi(t) \equiv \sigma \frac{\ddot{a}}{a} - \sigma(\sigma + 1) \left(\frac{\dot{a}}{a}\right)^2 + X.$$
(283)

In the case of the bouncing universe given by Eq. (181), we have

$$\left(t^{2} + \alpha_{0}^{2}\right)^{2} \ddot{g} + \left(\alpha t^{2} + \beta \alpha_{0}^{2}\right)g = 0,$$
(284)

where $\alpha \equiv k^2/3 - 7/4$ and $\beta \equiv k^2/3 - 1/2$. With the change of variable $z = 1/2 - it/(2\alpha_0)$, this equation takes the form

$$\frac{d^2g}{dz^2} + I(z)g = 0,$$
(285)

where

$$I(z) = -\frac{\beta}{4z^2 (z-1)^2} + \frac{\alpha (2z-1)^2}{4z^2 (z-1)^2}.$$
(286)

After a direct calculation, Eq. (285) can be transformed into a hypergeometric equation

$$z(1-z)\frac{d^{2}\omega}{dz^{2}} + [c - (a+b+1)z]\frac{d\omega}{dz} - ab\omega = 0,$$
(287)

where

 $a = \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha},\tag{288}$

$$b = \frac{1}{2} - \sqrt{\frac{1}{4} - \alpha},\tag{289}$$

$$c = \frac{5}{2}.$$
(290)

The solution for g(z) is given by

$$g(z) = z^{\frac{c}{2}} (z-1)^{\frac{c-a-b-1}{2}} \omega(z)$$
(291)

ot, in terms of the hypergeometric function F(a, b, c; z),

$$g(z) = z^{\frac{5}{4}} (z-1)^{-\frac{1}{4}} F\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}, \frac{1}{2} - \sqrt{\frac{1}{4} - \alpha}, \frac{5}{2}; z\right).$$
(292)

Finally, the solution for the electric part of the Weyl tensor, is given by

$$E_k = s(-4\alpha_0^2)^{-\frac{5}{4}} (z-1)^{-\frac{3}{2}} F\left(\frac{1}{2} + \sqrt{2 - \frac{k^2}{3}}, \frac{1}{2} - \sqrt{2 - \frac{k^2}{3}}, \frac{5}{2}; z\right)$$
(293)

where *s* is a constant. Restricting to $z \in \mathfrak{R}$ it follows that this solution is regular for z < 1, and can be analytically extended for all values of *z*. Hence, the perturbation is regular.

Notice that the power spectrum of perturbations can be obtained using (see Appendix)

$$P_k = k^{-1} |E_k|^2. (294)$$

The second example we shall study in this section is the model presented in Section 4.5.1, the perturbation of which was analyzed by the quasi-Maxwellian method in [326]. In this model, the singularity is avoided by the introduction of nonlinear corrections to Maxwell electrodynamics, given by

$$L = -\frac{1}{4}F + \alpha F^2 + \beta G^2,$$
(295)

194

¹⁰⁵ Notice that, as shown in the Appendix, this equation is actually a consequence of a transformation that takes the variables (E, Σ) (which are not canonically conjugated) into a new pair of variables that are canonically conjugated.

where $F = F_{\mu\nu}F^{\mu\nu}$, $G \doteq \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$, α and β are arbitrary constants. After an average procedure (see Section 4.5.1), the expression for the scale factor for the "magnetic unverse" with $\epsilon = 0$ is:

$$a(t)^{2} = \mathscr{H}_{0} \left[\frac{2}{3} (t^{2} + 12\alpha) \right]^{1/2}.$$
(296)

The interpretation of the source as a one-component perfect fluid in an adiabatic regime leads to instabilities [339], which are artificial, as will be seen next. The sound velocity of the fluid in this case is given by [249]

$$\frac{\partial p}{\partial \rho} = \frac{\dot{p}}{\dot{\rho}} = -\frac{\dot{p}}{\theta(\rho+p)}.$$
(297)

This expression, involving only background quantities, is not defined at the points where the energy density attains an extremum given by $\theta = 0$ and $\rho + p = 0$. In terms of the cosmological time, these points are determined by t = 0 and $t = \pm t_c = 12\alpha$. Notice that they are well-behaved regular points of the geometry, indicating that the occurrence of a singularity is in fact caused by an inappropriate description of the source. This difficulty can be circumvented by splitting the part coming from Maxwell's dynamics from the additional non-linear α -dependent term in the Lagrangian. As a result, we get two noninteracting perfect fluids:

$$T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)}, \tag{298}$$

where

 $\langle \mathbf{n} \rangle$

$$T_{\mu\nu}^{(1)} = (\rho_1 + p_1) \nu_{\mu} \nu_{\nu} - p_1 g_{\mu\nu}, \tag{299}$$

$$T^{(2)}_{\mu\nu} = (\rho_2 + p_2) \,\nu_\mu \nu_\nu - p_2 \,g_{\mu\nu},\tag{300}$$

and

$$\rho_1 = \frac{1}{2} \mathscr{H}^2,\tag{301}$$

$$p_1 = \frac{1}{c} \mathcal{H}^2, \tag{302}$$

$$\rho_2 = -4\alpha \,\mathscr{H}^4,\tag{303}$$

$$p_2 = -\frac{20}{3} \,\alpha \,\mathscr{H}^4. \tag{304}$$

From this decomposition it follows that each of the components of the fluid satisfies the conservation equation, thus showing that the source can be described by two non-interacting perfect fluids with equations of state $p_1 = 1/3 \rho_1$ and $p_2 = 5/3 \rho_2$. This splitting should be understood only as a mathematical device to allow for a fluid description.

From the considerations presented in Section 11.5.2 we obtain [326]:

$$\dot{\Sigma}_{1} = -\left(\frac{2\lambda_{1}(3\epsilon + k^{2})}{a^{2}(1+\lambda_{1})\rho_{1}} + 1\right)E_{1},$$
(305)

$$\dot{\Sigma}_2 = -\left(\frac{2\lambda_1(3\epsilon + k^2)}{a^2(1 + \lambda_2)\rho_2} + 1\right)E_2,$$
(306)

$$\dot{E_1} + \frac{1}{3}\theta E_1 = -\frac{1}{2} (1 + \lambda_1) \rho_1 \Sigma_1, \tag{307}$$

$$\dot{E}_2 + \frac{1}{2}\theta E_2 = -\frac{1}{2}(1+\lambda_2)\,\rho_2 \Sigma_2,\tag{308}$$

where k is the wave number. As shown in [318], scalar perturbations can be expressed in terms of the two basic variables E_i and Σ_i , and the corresponding equations can be decoupled. The result in terms of the E_i is

$$\ddot{E}_i + \frac{4+3\lambda_i}{3}\theta\dot{E}_i + \left\{\frac{2+3\lambda_i}{9}\theta^2\left(\frac{2}{3}+\lambda_i\right)\rho_i\frac{1}{6}(1+3\lambda_j)\rho_j - \frac{(3\epsilon+k^2)\lambda_i}{a^2}\right\}E_i = 0.$$
(309)

Note that in this expression there is no summation in the indices, and $j \neq i$, and $\lambda_i = (\frac{1}{3}, \frac{5}{3})$. In the first case the equation for the variable E_1 becomes

$$\ddot{E}_1 + \frac{5}{3}\theta \dot{E}_1 + \left[\frac{1}{3}\theta^2 - \rho_1 - \rho_2 - \frac{5}{3a^2}\right]E_1 = 0.$$
(310)

Let us analyze the behavior of perturbations in the neighborhood of the points where the energy density attains an extremum (i.e. the bounce and the point in which $\rho + p$ vanishes). The expansion of the equation of E_1 in the neighborhood of the bounce (at t = 0) up to second order, is given by:

$$\ddot{E}_1 + at\dot{E}_1 + (b + b_1 t^2) E_1 = 0, \tag{311}$$

where the constants *a* and *b* are defined as follows

$$a = \frac{5}{2t_c^2},\tag{312}$$

$$b = -\frac{k^2}{\sqrt{6}\mathcal{H}_0 t_c},\tag{313}$$

$$b_1 = -\frac{b}{2t_c^2} - \frac{3}{4t_c^4}.$$
(314)

Defining a new function f as

$$f(t) = E_1(t) \exp\left\{ \left(+\frac{a}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a^2}{4}} \right) t^2 \right\},$$
(315)

and introducing the coordinate ξ by

$$\xi = -it^2 \sqrt{b_1 - \frac{a^2}{4}},$$
(316)

we obtain for *f* the confluent hypergeometric equation [5]

 $\xi \ddot{f} + (1/2 - \xi)\dot{f} + ef = 0, \tag{317}$

where

$$e = \frac{i(b - a/2)}{4(b_1 - a^2/4)^{1/2}} - \frac{1}{2}.$$
(318)

The solution of this equation is given by

$$f(t) = A M \left(d, 1/2, -it^2 \sqrt{b_1 - \frac{a^2}{4}} \right),$$
(319)

where *A* is an arbitrary constant and $M(d, 1/2, \xi)$ is the confluent hypergeometric function, which is well-behaved in the neighborhood of the bounce. Hence the perturbation $E_1(t)$ is regular and given by

$$E_1(t) = A M\left(d, 1/2, -it^2 \sqrt{b_1 - \frac{a^2}{4}}\right) \exp\left\{\left(-\frac{a}{4} + \frac{i}{2} \sqrt{b_1 - \frac{a^2}{4}}\right) t^2\right\}.$$
(320)

After a similar procedure, the perturbation E_2 obeys, in the same neighborhood, the following equation:

$$\ddot{E}_2 + at\dot{E}_2 + (b + b_1 t^2) E_2 = 0.$$
(321)

This is the same equation we obtained for E_1 , with different values of a, b and b_1 given in this case by

$$a = \frac{9}{2t_c^2},\tag{322}$$

$$b = \frac{3}{2t_c^2} - 5\frac{k^2}{\sqrt{6}\mathcal{H}_0 t_c},$$
(323)

$$b_1 = -\frac{5k^2}{t_c^3 \mathscr{H}_0 \sqrt{6}} - \frac{5}{t_c^4}.$$
(324)

The solution is given by the real part of

$$E_2(t) = AM\left(d, 1/2, -it^2\sqrt{b_1 - \frac{a^2}{4}}\right) \exp\left\{-\left(\frac{a}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a^2}{4}}\right)t^2\right\},$$
(325)

$$\ddot{E}_1 + a\dot{E}_1 + (b + b_1 t) E_1 = 0, \tag{326}$$

with

$$a = \frac{5}{4t_c},\tag{327}$$

$$b = -\frac{3}{4t_c^2} - \frac{\sqrt{3k^2}}{6\mathscr{H}_0 t_c},$$
(328)

$$b_1 = \frac{\sqrt{3}}{4t_c^2} \left(\frac{k^2}{3\mathscr{H}_0} - \frac{3}{2t_c} \right).$$
(329)

By the following variable transformation:

$$E_1(t) = \exp\left(-\frac{at}{2}w(t)\right),\tag{330}$$

the differential equation goes to

$$\ddot{w} + \left(b - (a/2)^2 + b_1 t\right) w = 0, \tag{331}$$

and the solution is

0

$$w(t) = w_0 \operatorname{Ai}\left(-\frac{b - (a/2)^2 + b_1 t}{b^{2/3}}\right).$$
(332)

The Airy function Ai is regular near $t = t_c$, and so is E_1 . Finally we look for the equation of E_2 at the neighborhood of $t = t_c$:

$$\ddot{E}_2 + a\dot{E}_2 + (b + b_1 t)E_2 = 0, \tag{333}$$

where

$$a = \frac{9}{4t_c},\tag{334}$$

$$b = \frac{5}{t_c} \left(\frac{5}{4t_c} - \frac{\sqrt{3}m^2}{6\mathscr{H}_0} \right),$$
(335)

$$b_1 = \frac{5\sqrt{3}}{2t_c^2} \left(\frac{1}{t_c} - \frac{m^2}{6\mathscr{H}_0} \right).$$
(336)

This equation differs from Eq. (326) only by the numerical values of the parameters a, b, and b_1 so we obtain the same type of regular solution

$$E_2 = w_0 \operatorname{Ai}\left(-\frac{b - (a/2)^2 + b_1 t}{b_1^{2/3}}\right) \exp\left(-\frac{at}{2}\right).$$
(337)

Hence, it was shown by a direct analysis of a specific nonsingular universe, that in the neighborhood of the special points in which a change of regime occurs, all independent perturbed quantities are well-behaved, and the model is stable with regard to scalar perturbations.

A similar analysis has been carried out for the model described by Eq. (296) in the case of tensor perturbations in [19]. The result shows differences between gravitational waves generated near a singularity and those generated near the bounce. While in the first case the system exhibits a node–focus transition in the (E, Σ) plane, independently of the perturbation wavelength λ , in the bouncing model the trajectories may exhibit a focus–node–focus transition, or no transition at all, depending on the value of λ .

11.3. Matching

As mentioned in Section 11, another approach to the description of perturbations in a bouncing universe uses the idea of matching a contracting with an expanding phase (see for instance [128]). The hope here again resides in the fact that some general features can be extracted from given examples, since the matching may be done in such a way as to avoid a very detailed specification of the high curvature phase. Inasmuch as the result depends on the matching conditions, this issue was the subject of a long debate [272]. We shall present next some examples of this technique.

The case of a scalar field with an exponential potential (inspired in the string pre-big bang and the ekpyrotic model) was studied in [139]. A matching between a contracting, scalar field-dominated phase and an expanding, radiation-dominated phase (and also of the corresponding perturbations) was done using the Israel conditions [218]. It was assumed that the slice of spacetime in which high-energy physics takes control is very thin, and can be approximated by a spacelike surface, with a negative surface tension (to be specified by the underlying physics) required by the jump in the extrinsic curvature. Neglecting possible, but subdominant, anisotropic surface stresses, ¹⁰⁶ and depending on the chosen surface, it was found that a scale-invariant spectrum could be transferred from the contracting to the expanding phase. A similar model has been studied in [162], where it was shown that the value p = 2/3 of the power law $a(t) \propto (-t)^p$ was adopted for the scale factor generates a scale-invariant spectrum of adiabatic curvature fluctuations in the collapsing phase. The chosen background corresponds to a contracting Universe dominated by cold matter with null pressure. As a result of the glueing, the spectrum is matched at the bounce to a scale-invariant spectrum during the expanding phase. This model was also shown to generate a scale-invariant spectrum of gravitational waves, as already realized in [271].

It is useful to assume that the physics of the bounce is encoded in the transfer matrix T, defined by

$$\binom{D_{+}}{S_{+}} = \binom{T_{11} \quad T_{12}}{T_{21} \quad T_{22}} \binom{D_{-}}{S_{-}} .$$
(338)

T gives the degree of mixing between the dominant (D) and sub-dominant (S) modes before and after the bounce for a fixed comoving wave number *k*. Several combinations are possible, such as one for which the spectrum is initially not scale invariant but is turned into it because of a nontrivial *k* dependence of the transition matrix. Due to the fact that the bounce lasts only a short time, it is conceivable that it does not exert any influence on the large scales that are of astrophysical interest today. This implies that *T* does not depend on *k* [139], in such a way that a scale invariant pre-bounce spectrum is transmitted without change to the post-bounce phase. This hypothesis has been tested in [280]. It was shown by way of an example (a bouncing solution in general relativity, with positive curvature spatial section, with a scalar field as a source, by using an expansion of the bouncing scale factor around the $\epsilon = 1$ de Sitter-like bouncing solution) that *T* may depend on *k*, provided that the null energy condition (NEC) is very close to being violated at the bounce, hence affecting the large scale behaviour of scalar perturbations (see however [129,282]). Note however that it was shown in [280] that the spectrum of gravitational waves is not affected by the bounce.

The authors of [105] have obtained the most general form of the transfer matrix respecting local causality. In particular, they have shown that no local-causality-respecting matching condition can lead to a scale invariant spectrum for both the pre-big-bang and the ekpyrotic model, in agreement with the result of [118]. They also studied a non-local model based on string theory and showed that under certain conditions a post-bounce SIS is possible.

A different line of attack was pursued in [73] with the central assumption that the bounce in a spatially flat universe is governed by just one physical scale (chosen as η_B , the cosmological time at which the bounce occurs). Working in GR and incorporating all the eventual new physics in the matter side of EE, the general solution to the problem of the propagation of perturbations through the bounce was presented in [73]. It was shown that the spectrum of the Bardeen potential in the expansion phase depends critically on the relation between the comoving pressure perturbation and the Bardeen potential in the new physics sector of the energy–momentum tensor. Only if the comoving pressure perturbation is directly proportional to the Bardeen potential (rather than its Laplacian, as for any known form of ordinary matter), the pre-bounce growing mode of the Bardeen potential persists in the post-bounce constant mode. This would open the door to models with a scale-invariant spectrum (hence in agreement with observations) for those cases in which there is very slow contraction in the pre-bounce. This result is supported by numerical analysis of a toy model in which $\delta p \propto \Psi$ [73]. Examples of this type of behaviour for perturbations are given by models with spatial curvature (which cannot be treated however with this approach) and also by models with modifications coming from extra dimensions (such as the one presented in [41]) [73].

11.4. Creation of cosmological magnetic fields

The origin, evolution, and structure of large-scale magnetic fields are amongst the most important issues in astrophysics and cosmology. The standard model for the generation of this fields is the dynamo, which amplifies a small seed field to the current observed values of $1 - \text{few } \mu G$. There are several mechanisms to produce these seeds, but the prevalent view is that they have a primordial origin [190]. In particular, vacuum fluctuations of the EM field may be "stretched" by evolution of the background geometry to super-horizon scales, and they could appear today as large-scale EM fields. For this to happen, conformal invariance of the EM equations must be broken. This is the case in models such as dilaton electrodynamics [255] and Weyl integrable spacetime (see Section 3.2 and [365] for a list of references on the subject).

As a previous step in the details of the case of the EM field, let us discuss the creation of massive scalar particles in a bouncing universe with $\epsilon = -1$, following [115]. The expansion factor is given by $a(t) = t^2 + a_0^2$, or $a(\eta) = a_0^2 \cosh \eta$ in conformal time, as in the examples studied in [286,310]. The EOM for the scalar field is

$$\Box \phi + \left(m^2 + \frac{1}{6}\xi R\right)\phi = 0.$$

¹⁰⁶ This restriction was lifted in [113].
With the mode decomposition

$$\phi_k(\mathbf{x}) = a(\eta)^{-1/2} Y_{\mathbf{k}}(\vec{\mathbf{x}}) \chi_k(\eta),$$

where $\mathbf{k} = (k, J, M)$ and the $Y_{\mathbf{k}}(\vec{x})$ are given in terms of the spherical harmonics (see [56]), the function $\chi_k(\eta)$ satisfies the modified Mathieu equation:

$$\frac{\mathrm{d}^2\chi_k}{\mathrm{d}\eta^2} - (\lambda - 2h^2\cosh^2\eta)\chi_k = 0,$$

where $\lambda \equiv -(k^2 + \frac{1}{2}m^2a_0^2)$, and $h \equiv \frac{1}{2}ma_0$. The number of created quanta in the (asymptotically flat) future can be calculated with the solutions of this equation that have the right asymptotic behaviour, and following standard techniques. In the limit $h \ll 1$ (*i.e* when the Compton wavelength of the particle is much greater than a_0), the result is [115]

$$|\beta_k|^2 = \frac{1}{2\sinh^2 \pi \tilde{k}} \left[1 - \cos\left(4\tilde{k}\ln\frac{h}{2}\right) + \varphi \right],$$

where \tilde{k} is the index in the Mathieu functions $M_{-i\tilde{k}}(\eta, h)$, and is a complicated function of λ and h, which in the limit for small h reduces to

$$\lambda = -\tilde{k}^2 - \frac{h^4}{2(\tilde{k}+1)} + O(h^8),$$

and φ is a phase, independent of *h*. The expression for $|\beta|$ varies from 0 to $4 \times \exp(-2\pi \tilde{k})$ for large *k*, and shows that for a given *k*, the particle number depends on the product ma_0 .

The creation of magnetic fields in a bouncing universe in models that break conformal invariance with a coupling to a scalar field was studied in [364,185,365]. In the latter, canonical quantization was applied to the model given by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} f(\omega) F_{\alpha\beta} F^{\alpha\beta},$$

where ω is the scalar field, and $F_{\alpha\beta}$ an abelian field, with $f(\omega) = \exp(-2\omega)$. The modes of the potential $A_{\mu} = e^{-\omega}A_{\mu}$ satisfy the equation

$$\mathcal{A}_{k\alpha}^{\prime\prime\sigma}(\eta) + (k^2 - V(\eta))\mathcal{A}_{k\alpha}^{(\sigma)},\tag{339}$$

where $\sigma = +, -$ designates the base of travelling waves, $\alpha = 1, 2$ describes the two transverse degrees of freedom, and $V(\eta) = -\omega'' + {\omega'}^2$. For the background described in Eq. (118),

$$V(\eta) = \frac{2\sigma \sinh(2\eta) + \sigma^2}{\cosh^2(2\eta)},\tag{340}$$

where $\sigma \equiv \sqrt{6}/\lambda$, where λ^2 is the coupling constant of the scalar field to gravity (see Fig. 16). The mode equation (339) admits analytical solutions in terms of hypergeometric functions, in terms of which the Bogolubov coefficients, and the expression for the energy density of the magnetic field ρ_m can be calculated [365]. The amplification factor with respect to the conformal vacuum peaks for the modes with momenta such that $k \approx 1.31$, and is given by

$$\frac{\rho_m}{(\rho_m)_{cf}} \propto \exp\left(\frac{\pi\sqrt{6}}{\lambda}\right),\tag{341}$$

for $\eta \gg 1$. The conditions for the spectrum to be greatly amplified today are [365]

$$a_0 \ll ct_r, \qquad \lambda \ll 1,$$

where t_r is the time at which the scalar field is negligible, in such a way that the EM field is free again.

At a comoving scale of about 10 kpc, the strength of conformal vacuum fluctuations is of the order of 10^{-55} G. To reach the strength required to feed the galactic dynamo, $B_{\text{seed}} \propto 10^{-20}$ G, which is a conservative estimate, we get from Eq. (341) that $\lambda \approx 0.1$. Taking for the comoving scale the size of the universe ($\approx 4 \times 10^6$ kpc), the amplification factor becomes 10^{46} , and we need $\lambda \approx 0.07$. So the strength needed in both cases can be achieved by a modest value of λ , the coupling constant of ω to gravity.

These results were obtained in a model that did not take into account the effect of the creation of matter by the decay of the scalar field. The solution presented in Section 3.2.9, namely

$$a(\eta) = \beta_{\sqrt{\cosh(2\eta)} + k_0 \sinh(2\eta) - 2k_0 (\tanh \eta + 1)},$$
(342)

with $\beta = a_0/\sqrt{1 - k_0}$, and $0 < k_0 < 1/7$ incorporates this feature, and its influence on the creation of photons was discussed in [364]. The result, displayed in Fig. 17, shows that there is a substantial increment in the number of photons if we take into account the effect of matter creation.



Fig. 16. Plot of the potential (see Eq. (340)), for $\lambda = 0.1, 0.3, 0.5$ (solid, dotted, dashed line respectively).



Fig. 17. Plot of the mean number of photons as a function of the conformal time for m = 20, in the case without matter (dashed line) and for the case with matter creation (full line), for $\lambda = 1$.

11.5. Appendix

In this appendix, we give a short summary of two gauge-invariant methods that can be applied to study perturbations in cosmological scenarios.

11.5.1. Perturbations using Bardeen variables

Fluctuations of the metric tensor can be classified by their properties under spatial rotations into scalar, vector and tensor perturbations. In the linear theory, their evolution is decoupled. In the case of scalar perturbations, the perturbed metric of a homogeneous and isotropic spacetime can be written as

$$ds^{2} = a^{2}(\eta) \left\{ (1 + 2\phi) d\eta^{2} - 2B_{;i} d\eta dx^{i} - [(1 - 2\psi)\gamma_{ij} + 2E_{,i;j} dx^{i} dx^{j}] \right\},$$
(343)

where γ_{ij} is the metric of the 3-space. We shall sketch the case of hydrodynamical perturbations of a perfect fluid¹⁰⁷ with energy–momentum tensor

$$T^{\alpha}_{\ \beta} = (\rho + p)u^{\alpha}u_{\beta} - p\delta^{\alpha}_{\ \beta}. \tag{344}$$

¹⁰⁷ For other cases, such as s scalar field, see [293].

Following [28], it is convenient to build, from the four variables appearing in (343), two gauge-invariant quantities, given by

$$\Phi = \phi + \frac{\left[(B - E')a'\right]'}{a}, \qquad \Psi = \psi - \frac{a'(B - E')}{a}.$$

In terms of these, the gauge-invariant perturbed EE are

$$-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2 \Psi + 3k\Psi = \frac{1}{2}a^2 \delta T_0^{(gi)0},$$
(345)

$$(\mathcal{H}\Phi + \Psi')_{,i} = \frac{1}{2}a^2 \delta T_i^{(gi)0}, \tag{346}$$

$$\left[2\mathcal{H}' + \mathcal{H}^{2}\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - k\Psi + \frac{1}{2}\nabla^{2}(\Phi - \Psi)\right]\delta_{j}^{i} - \frac{1}{2}\gamma^{ij}(\Phi - \Psi)_{|kj} = -\frac{1}{2}a^{2}\delta T_{j}^{(gi)i},$$
(347)

where the $\delta T^{(gi)\alpha}_{\beta}$ are gauge invariant combinations of the δT^{μ}_{ν} , *B*, and *E* (see [293] for details). In the case of hydrodynamical matter, the most general form of the perturbation can be written in terms of the perturbed energy $\delta \rho$, the perturbed pressure δp , the potential \mathcal{V} of the 3-velocity $v^i(t, \vec{x})$, and the anisotropic stress σ as follows [28]:

$$(\delta T^{\mu}_{\nu}) = \begin{pmatrix} \delta \rho & -(\rho_0 + p_0)a^{-1}\mathcal{V}_{,i} \\ (\rho_0 + p_0)a\mathcal{V}_{,i} & -\delta p\delta_{ij} + \sigma_{ij} \end{pmatrix}$$

For the case of a perfect fluid, with energy–momentum tensor given by Eq. (344), $\sigma_{ii} = 0$. The pressure perturbation can be split into its adiabatic and entropy components as

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{S} \delta \rho + \left(\frac{\partial p}{\partial S}\right)_{\rho} \delta S \equiv c_{s}^{2} \delta \rho + \tau \delta S.$$
(348)

Entropy perturbations may be important in the case of two-component systems, such as plasma and radiation.

Gauge-invariant perturbations of the energy-momentum tensor can be expressed in terms of the gauge-invariant energy density, pressure, and velocity perturbation:

$$\delta T_0^{(gi)0} = \delta \rho^{(gi)}, \qquad \delta T_i^{(gi)0} = (\rho_0 + p_0)a^{-1}\delta u_i^{(gi)}, \qquad \delta T_j^{(gi)i} = -\delta p^{(gi)}\delta_j^i,$$

with

$$\delta \rho^{(gi)} = \delta \rho + \rho'_0(B - E'), \qquad \delta p^{(gi)} = \delta p + p'_0(B - E'), \qquad \delta u_i^{(gi)} = \delta u_i + a(B - E')_{|i|}.$$

From Eqs. (345) to (347) applied to this case, it follows that $\Phi = \Psi$. Using Eq. (348), the system can be written as

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1+3c_s^2)(\mathcal{H}^2 - k)]\Phi = \frac{1}{2}a^2\tau\delta S.$$
(349)

$$(a\Phi)'_{,i} = \frac{1}{2}a^2(\rho_0 + p_0)\delta u_i^{(gi)}.$$
(350)

For adiabatic perturbations, Eq. (349) yields Φ , which determines $\delta \rho^{(gi)}$ through Eq. (345), and $\delta u_i^{(gi)}$ through Eq. (350). Eq. (349) can be simplified with the change of variables

$$\Phi = \sqrt{\frac{1}{2}} \sqrt{\frac{\mathcal{H}^2 - \mathcal{H}' + k}{a^2}} u,$$

vielding

$$u''-c_s^2\nabla^2 u-\frac{\theta''}{\theta}\ u=\mathcal{N},$$

with

$$\begin{split} \theta &= \frac{1}{a} \left(\frac{\rho_0}{\rho_0 + p_0} \right)^{1/2} \left(1 - \frac{3\epsilon}{a^2 \rho_0} \right)^{1/2}, \\ \mathcal{N} &= a^2 (\rho_0 + p_0)^{-1/2} \tau \delta S. \end{split}$$

Vector perturbations

The most general perturbed metric including only vector perturbations is given by¹⁰⁸

$$(\delta g_{\mu\nu}) = \begin{pmatrix} 0 & -S^i \\ -S^i & F^i_{,j} + F^j_{,i} \end{pmatrix},$$

¹⁰⁸ The results quoted in this section are taken from [293].

where the vectors S and F are divergenceless. From their transformation properties, it can be shown that

$$\sigma^i = S^i + \dot{F}^i$$

(where the dot means derivative w.r.t. conformal time) is a gauge-invariant quantity. For the perturbations of the stressenergy tensor, we have

$$(\delta T^{\alpha}_{\beta}) = \begin{pmatrix} 0 & -(\rho_0 + p_0)V^i \\ (\rho_0 + p_0)(V^i + S^i) & p_0(\pi^i_{,j} + \pi^j_{,i}) \end{pmatrix},$$

where V^i and π^i are divergenceless. V^i is related to the perturbation of the 4-velocity by

$$(\delta u^{\mu}) = \begin{pmatrix} 0\\ V^i/a \end{pmatrix}.$$

The gauge-invariant quantities are given in this case by $\theta^i = V^i - \dot{F}^i$ and π^i . Adopting the Newtonian gauge (in which F = 0), from the perturbed EE we get

$$-\frac{1}{2a^2}\nabla^2 S^i = (\rho + p)V^i,$$

$$-\frac{1}{2a^4}\nabla_t (a^2(S^{j}_{,i} + S^{i}_{,j})) = p(\pi^i_{,j} + \pi^{j}_{,i}),$$
(351)
(352)

where ∇^2 is the spatial Laplacian. From Eq. (351) we get

$$V_k^i = \frac{1}{2a^2(\rho+p)}k^2 S_k^i,$$
(353)

for the Fourier modes of V and S. Assuming that there are no anisotropic stresses, as in the case of pressureless dust, we get from Eq. (352),

 $\nabla_t (a^2 S^i_{\ \nu}) = 0.$

Hence $S_k^i = C_k^i/a^2$, where C is a constant. From this and Eq. (353), we get

$$V_k^i \propto \frac{k^2 C_k^i}{a^{1-3\omega}}.$$
(354)

Note that V_k^i increases for $\omega = 0$, and stays constant for radiation, but S_k^i always increases for decreasing *a*.

Tensor perturbations

These perturbations are built using a symmetric 3-tensor h_{ij} which satisfies the constraints

$$h_{i}^{i} = 0$$
 $h_{ii}^{[j]} = 0$

in such way that the metric for tensor perturbations is

$$(\delta g_{\mu\nu}^{(t)}) = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}.$$

From the perturbed EE we find (see for instance [294])

$$h_{ij}^{\prime\prime} + 2\mathcal{H}h_{ij}^{\prime} - \bigtriangleup h_{ij} = 2a^2 \delta T_{ij}^{(gi)T},$$

where $\delta T_{ij}^{(gi)T}$ is the gauge-invariant "pure tensor" part of $\delta T_{\mu\nu}$. In Fourier space, and introducing the rescaled variable $h_{ij} = e_{ij}v/a$, we have

$$v^{\prime\prime} + \left(k^2 - \frac{a^{\prime\prime}}{a}\right)v_k = 0.$$
(355)

11.5.2. The quasi-Maxwellian method

The OM method has its roots in the formulation of Jordan and his collaborators [225] and uses Bianchi identities to propagate initial conditions. The basic idea is to identify gauge invariant quantities from the beginning, using Stewart's lemma [388], which guarantees that the perturbation of an object Q is gauge-invariant if Q is either constant or a linear combination of δ^{μ}_{ν} with constant coefficients. In conformally flat models, the Weyl tensor (defined below) is identically zero, so its perturbation is a true perturbation, and not a gauge artifact. We shall see below how to obtain a minimum set of variables to completely characterize a perturbation, along with their evolution equations.

Definitions and notation

The Weyl conformal tensor is defined by means of the expression

 $W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6}Rg_{\alpha\beta\mu\nu},$

u v

where

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}, \tag{356}$$

and

_

$$2M_{\alpha\beta\mu\nu} = R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}.$$
(357)

The 10 independent components of the Weyl tensor can be separated in the electric and magnetic parts, defined (in analogy with the electromagnetic field) for an observer with 4-velocity v^{μ} as:

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu}v^{\mu}v^{\nu}, \tag{358}$$

$$H_{\alpha\beta} = -W^*_{\alpha\mu\beta\nu}v^{\mu}v^{\nu}.$$
(359)

The dual operation was performed with the completely skew-symmetric Levi–Civita tensor $\eta_{\alpha\beta\mu\nu}$. From the symmetry properties of the Weyl tensor it follows that the operation of taking the dual is independent on the pair in which it is applied. It follows from these definitions that the tensors $E_{\mu\nu}$ and $H_{\mu\nu}$ are symmetric, traceless and belong to three-dimensional

space orthogonal to the observer with 4-velocity v^{μ} , that is:

$$E_{\mu\nu} = E_{\nu\mu}, \qquad E_{\mu\nu}v^{\mu} = 0, \qquad E_{\mu\nu}g^{\mu\nu} = 0, \tag{360}$$

and similar relations for $H_{\mu\nu}$. The metric $g_{\mu\nu}$ and the vector v^{μ} (tangent to a timelike congruence of curves Γ) induce a projector tensor $h_{\mu\nu}$ which separates any tensor in terms of quantities defined along Γ plus quantities defined on the 3dimensional space orthogonal to v^{μ} . The tensor $h_{\mu\nu}$, defined on this 3-dimensional space is symmetric and a true projector, that is

$$h_{\mu\nu}h^{\nu\lambda} = \delta_{\mu}{}^{\lambda} - \nu_{\mu}\nu^{\lambda} = h_{\mu}{}^{\lambda}. \tag{361}$$

We shall work with the FLRW geometry written in the standard Gaussian coordinate system:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{362}$$

where $g_{ij} = -a^2(t)\gamma_{ij}(x^k)$. 3-dimensional geometry has constant curvature and thus the corresponding Riemannian tensor $^{(3)}R_{iikl}$ can be written as

$$^{(3)}R_{iikl} = \epsilon \gamma_{iikl}$$

The covariant derivative in the 4-dimensional space-time will be denoted by the symbol ";" and the 3-dimensional derivative will be denoted by "||".

The irreducible components of the covariant derivative of v^{μ} are given in terms of the expansion scalar (θ), shear ($\sigma_{\alpha\beta}$), vorticity $(\omega_{\mu\nu})$ and acceleration (A_{α}) by the standard definition:

$$\nu_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta} + \omega_{\alpha\beta} + A_{\alpha}\nu_{\beta}, \tag{363}$$

where

$$\sigma_{\alpha\beta} = \frac{1}{2} h^{\mu}_{(\alpha} h^{\nu}_{\beta)} v_{\mu;\nu} - \frac{1}{3} \theta h_{\alpha\beta},$$

$$\theta = v^{\alpha}_{;\alpha},$$

$$\omega_{\alpha\beta} = \frac{1}{2} h^{\mu}_{[\alpha} h^{\nu}_{\beta]} v_{\mu;\nu},$$

$$A_{\alpha} = v_{\alpha;\beta} v^{\beta}.$$

(364)

We also define

$$\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}.$$
(365)

Quasi-Maxwellian equations of gravity and their perturbation

We shall present in this subsection a sketch of the deduction of the equations that govern the perturbations in quasi-Maxwellian formalism. Details of the calculations in this section can be found in [318]. Using Einstein's equations and the definition of Weyl tensor, Bianchi identities can be written in an equivalent form as

$$\begin{split} W^{\alpha\beta\mu\nu}{}_{;\nu} &= \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha} R^{,\beta]} \\ &= -\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha} T^{,\beta]}. \end{split}$$

Quasi-Maxwellian equations of gravity are obtained by projecting these equations (i.e. Bianchi identities are taken as true dynamical equations which describe the propagation of gravitational disturbances). The evolution equation for the perturbations for $\delta\theta$, $\delta\sigma_{\mu\nu}$, and $\delta\omega^{\mu}$, as well as 3 constraint equations, are obtained projecting and perturbing the equation

$$v_{\mu;\alpha;\beta} - v_{\mu;\beta;\alpha} = R_{\mu\omega\alpha\beta}v$$

which follows from the definition of the curvature tensor. Finally we get two more equations by projecting the conservation $\log T^{\mu\nu}_{;\nu} = 0$. Adding up, we have a set of twelve equations which when perturbed yield (after straightforward manipulations) the coupled differential equations needed to give a complete description of the perturbation. In a general case, the variables are

$$\mathcal{V} = \{\delta E_{ii}, \delta H_{ii}, \delta \omega_{ii}, \delta \sigma_{ii}, \delta \pi_{ii}, \delta A_i, \delta q_i, \delta \rho, \delta \theta, \delta V_0, \delta V_k\},\$$

where δq_i is the perturbation of the heat flux. From now on we will concentrate on the case of scalar irrotational perturbations. As shown in [259], it is useful to develop the perturbed quantities in the spherical harmonics basis. It is enough for our purposes to work only with scalar quantities, denoted by $Q^{(k)}(x^i)$ (with $\partial Q^{(k)}/\partial t = 0$) and the vector and tensor quantities that follow from it, defined by $Q_i^{(k)} \equiv Q_{i}^{(k)}$, $Q_{ij}^{(k)} \equiv Q_{ij}^{(k)}$. The scalar $Q^{(k)}$ obeys the eigenvalue equation defined in the 3-dimensional background space by:

$$\nabla^2 Q^{(k)} = kQ^{(k)},$$
(366)

where *k* is the wave number, and the symbol ∇^2 denotes the 3-dimensional Laplacian:

$$\nabla^2 Q \equiv \gamma^{ij} Q_{,i|j} = \gamma^{ij} Q_{,i;j}. \tag{367}$$

Since the modes do not mix at the linear order, we will drop the superindex (k) from Q. The traceless operator \hat{Q}_{ij} is defined as

$$\hat{Q}_{ij} = Q_{ij} + \frac{k^2}{3} Q \gamma_{ij}, \tag{368}$$

and the divergence of \hat{Q}_{ii} is given by

$$\hat{Q}_{ij}^{ij} = -2\left(\epsilon + \frac{k^2}{3}\right)Q^i.$$
(369)

Due to Stewart's lemma, the good (since they are gauge-invariant and null in the background) objects in the list \mathcal{V} are δE_{ij} , $\delta \pi_{ij}$, δa_i , and δq_i . According to causal thermodynamics the evolution equation of anisotropic pressure is related to the shear through [219]

$$\tau \dot{\Pi}_{ij} + \Pi_{ij} = \xi \sigma_{ij} \tag{370}$$

in which τ is the relaxation parameter and ξ is the viscosity parameter. For simplicity we will take the case in which τ can be neglected and ξ is a constant.¹⁰⁹ Eq. (370) then gives

$$\Pi_{ij} = \xi \sigma_{ij},\tag{371}$$

(372)

and the associated perturbed equation is

$$\delta \Pi_{ij} = \xi \delta \sigma_{ij}$$

We shall decompose the four independent and gauge-invariant perturbations as¹¹⁰

$$\begin{split} \delta E_{ij} &= \sum_{k} E^{(k)}(t) \hat{Q}_{ij}^{(k)}, \\ \delta \Sigma_{ij} &= \sum_{k} \Sigma^{(k)}(t) \hat{Q}_{ij}^{(k)}, \end{split}$$

¹⁰⁹ In the general case ξ and τ are functions of the equilibrium variables, for instance ρ and the temperature T and, since both variations $\delta \pi_{ij}$ and $\delta \sigma_{ij}$ are expanded in terms of the traceless tensor \hat{Q}_{ij} , it follows that the above relation does not restrain the kind of fluid we are examining. However, if we consider ξ as time-dependent, the quantity $\delta \pi_{ij}$ must be included in the fundamental set $\mathcal{M}_{[A]}$.

¹¹⁰ In fact, $\sqrt{\delta E_{ii} \delta E^{ij}}$ is the only quantity that characterizes without ambiguity a true perturbation of the Debever invariants [318].

$$\delta A_i = \sum_m \psi^{(m)}(t) Q_i^{(m)},$$

$$\delta q_i = \sum_m q^{(m)}(t) Q_i^{(m)}.$$

It can be shown that ψ is a function of Σ and E [318]. It follows that, restricting to the case q = 0 (no energy flux),¹¹¹ E(t) and $\Sigma(t)$ constitute the fundamental pair of variables in terms of which the dynamics for the perturbed FLRW geometry is completely characterized. Indeed, the evolution equations for these two quantities (which follow from Einstein's equations) generate a dynamical system involving only E and Σ (and background quantities) which, when solved, contains all the necessary information for a complete description of all remaining perturbed quantities of the FLRW geometry.

The evolution equations are given by [318]

$$\dot{\Sigma} = -E - \frac{1}{2} \xi \Sigma - k^2 \psi, \tag{373}$$

$$\dot{E} = -\frac{(1+\lambda)}{2}\rho\Sigma - \left(\frac{\theta}{3} + \frac{\xi}{2}\right)E - \frac{\xi}{2}\left(\frac{\xi}{2} + \frac{\theta}{3}\right)\Sigma - \frac{k^2}{2}\xi\psi.$$
(374)

As mentioned before, ψ can be expressed in terms of *E* and Σ^{112} :

$$(1+\lambda)\rho\psi = 2\left(1+\frac{3\epsilon}{k^2}\right)a^{-2}\left[-\lambda E + \frac{1}{2}\lambda\xi\Sigma + \frac{1}{3}\xi\Sigma\right].$$
(375)

Thus the set of perturbed equations reduces to a time-dependent dynamical system in the variables E and Σ :

$$\dot{\Sigma} = F_1(\Sigma, E), \qquad \dot{E} = F_2(\Sigma, E),$$
(376)

with

$$F_1 \equiv -E - \frac{1}{2}\xi \Sigma - k^2 \psi, \tag{377}$$

and

$$F_{2} \equiv -\left(\frac{1}{3}\theta + \frac{1}{2}\xi\right)E - \frac{k^{2}}{2}\xi\psi - \left(\frac{1}{4}\xi^{2} + \frac{(1+\lambda)}{2}\rho + \frac{1}{6}\xi\theta\right)\Sigma$$
(378)

where ψ is given in terms of *E* and Σ by Eq. (375), so the system (376) can be written as

$$\begin{pmatrix} \dot{E} \\ \dot{\Sigma} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E \\ \Sigma \end{pmatrix},$$
(379)

where

$$\alpha \equiv -\frac{\theta}{3}, \qquad \beta \equiv -\frac{1+\lambda}{2}\rho, \qquad \delta = 0, \qquad \gamma = \frac{6\lambda}{1+\lambda}\left(\epsilon + \frac{k^2}{3}\right)\frac{1}{a^2\rho} - 1.$$

Since

$$\frac{\partial \dot{E}}{\partial E} + \frac{\partial \dot{\Sigma}}{\partial \Sigma} = -\frac{\theta}{3},$$

the system (379) is not Hamiltonian due to the expansion of the universe. Nonetheless, new variables (Q, P) can be introduced in such a way that the system (379) is Hamiltonian. Defining

$$Q \equiv a^m \sigma, \qquad P = a^n E,$$

it is easily shown from the Poisson brackets that the otherwise arbitrary powers m and n must satisfy the relation m + n = 1 for the variables Q and P to be canonically conjugated. It follows that

$$\ddot{P} = \mathcal{M}_1 P + \mathcal{M}_2 Q.$$

The choice $n = 3\lambda/2 + 2$ yields $M_2 = 0$, and *P* satisfies the equation

$$\ddot{P} + \mu(t)P = 0,$$

¹¹¹ We further assume an equation of state relating the pressure and the energy density, i.e. $p = \lambda \rho$, which is preserved under arbitrary perturbations. ¹¹² Except when $(1 + \lambda) = 0$, see [318] for this case.

with

$$\mu(t) = \left(\frac{5}{4}\lambda + \frac{2}{3}\right)\rho + \frac{1}{a^2}\left[\frac{3\lambda}{2}\left(\frac{3\lambda}{2}\right)\epsilon - \lambda k^2\right],$$

which is equivalent to Eq. (282).

This method can be extended to vector and tensor perturbations in the FLRW model [319]. In the first case, the observable quantities are described in terms of the vorticity and the shear, while the electric and magnetic parts of the Weyl tensor suffice for the gravitational waves.¹¹³ The three types of perturbation are describable in Hamiltonian form, thus paving the way to canonical quantization [320], which was performed for scalar, vectorial, and tensor perturbations using the squeezed states formalism in [320]. In fact, in the case of scalar perturbations, the Hamiltonian in terms of the (*Q*, *P*) variables (with the choice m = 0) is given by

$$H = \frac{h_1}{2}Q^2 + \frac{h_2}{2}P^2 + \frac{h_3}{Q}P,$$

with

$$h_1 = rac{1+\lambda}{2}rac{
ho}{a}, \qquad h_2 = rac{6\lambda}{1+\lambda}\left(\epsilon + rac{k^2}{3}
ight)rac{1}{a
ho} - a, \qquad h_3 = 0.$$

11.6. Relation between the two methods

The Bardeen variables (Φ , Ψ) are related to the quasi-Maxwellian variables (*E*, Σ). For instance, in the case of scalar perturbations the relation between *E* and Φ (for a perfect fluid) is given by [189,224]

$$E=-k^2\Phi,$$

from which the relation for the spectrum given in Eq. (294) follows.

12. Conclusion

The idea of a bouncing universe has been considered since the early days of relativistic cosmology, as shown in this review. However, only a few analytical solutions describing a nonsingular universe served as a starting point to build a complete cosmological scenario. The main reason for this neglect by the majority of the physics community in the last 30 years of the 20th century was the strong influence of the singularity theorems, which led to the belief that some sort of singularity was inevitable in gravitational processes. The situation should have changed with the recently discovered positive acceleration of the universe since, in the realm of GR, accelerated expansion means that the matter content must satisfy the condition $\rho + 3p < 0$, which is precisely one of the conditions needed to have a bounce in Einstein's gravity. This violation of the SEC was already accepted in the early 80's in order to have a phase of inflationary expansion, and nowadays several systems are known which do not satisfy the inequality $\rho + 3p > 0$ (see for instance [27]). Hence, there is mounting evidence against one of the main theoretical prejudices forbidding bouncing universes in GR. Surprisingly, nonsingular models have not attracted the interest that should be expected based on the preceding considerations.¹¹⁴

Almost contemporaneous to the discovery of accelerated expansion was the gradual advent of a handful of cosmological models based on nonsingular solutions. These models aimed at solving the most stringent problems of the (pre-inflationary) cosmological standard model: the initial singularity, the isotropy and homogeneity of the currently-observed universe, the horizon problem, the flatness problem, ¹¹⁵ and the formation of structure. ¹¹⁶ Bouncing universes have partially met these challenges. The singularity is obviously absent, and its avoidance requires any of the assumptions listed in Section 1.2, which range from the violation of SEC (in GR) to quantum gravitational effects.

As explained in Section 1, a phase of accelerated contraction may solve the flatness problem in GR, and may also get rid of particle horizons (see for instance [179]).¹¹⁷

Finally, the amplification of primordial seeds (a problem prior to the formation of structure) in bouncing universes has been intensely debated recently (see Section 11). The asymptotic behavior of these universes is markedly different from that of the SCM or inflation. The universe at past infinity starts to collapse from a flat empty structure-less state that at

206

¹¹³ Perturbations in the Kasner solution were studied in [321].

¹¹⁴ It may be argued that this lack of interest is due to the fact that the bounce is expected to involve scales where quantum effects render GR inapplicable. But this is true also of the singularity theorems, as was known already in the early 70's. Moreover, there is no evidence against the possibility of a bounce in the classical regime [315], as follows from some of the models presented in Section 11.1, see also [158].

¹¹⁵ Note that the flatness problem may in principle not be a problem in gravitational theories other than GR (see Section 2.2).

¹¹⁶ In spite of its historical importance, the so-called monopole problem is not included in this list, since there is still room for it to be be considered as a problem of field theory first, and then (perhaps) of the standard cosmological model, see for instance [264,122].

¹¹⁷ See however the concerns in [89] about the efficiency of some bouncing models in erasing possible initial inhomogeneities.

past infinity can be approximated by Minkowski geometry written in terms of Milne coordinates.¹¹⁸ The transmission of the quantum fluctuations from this initial state to the post-bounce phase is strongly model-dependent, but there are some models which yield a scale-invariant spectrum for scalar perturbations in the post-bounce phase (see Section 11.1.1).

Offsprings of bouncing models are cyclic universes (see Section 10). Cyclic models also attempt to solve the abovementioned problems, and also may offer a new view on initial conditions: since by definition, there is neither a beginning nor an end of time in these models, there is no need to specify initial conditions. Generically, cyclic universes share the problems of the universes that bounce only once. In addition, they must assure that the large scale structure present in one cycle (generated by the quantum fluctuations in the preceding cycle) is not endangered by perturbations or structure generated in earlier cycles, and will not interfere with structure generated in later cycles. One of the latest cyclic models, presented in [155], claims to have successfully faced these issues (however see [265]).

As compelling a scenario may (or may not) seem, the ultimate judge is observation, so we can ask if there are any that may point to the occurrence of a bounce. As far as we know, there are two possibilities¹¹⁹:

- As discussed in Section 11.1, the tensor spectrum of a nonsingular universe has a unique feature. As an example, SPBB models predict a stochastic spectrum of gravitational waves whose amplitude increases as a function of frequency in some frequency ranges (see Section 11.1.3), hence avoiding the bounds due to the CMB, pulsar timing, and Doppler tracking [276]. The parameter space of the "minimal" SPBB model [175] was limited using LIGO results in [276]. Notice also that nonsingular universes may produce vector perturbations (see Section 11.1).
- The bounce may cause oscillations, that will be superimposed on the power spectrum of scalar perturbations. These oscillations would also appear in the WMAP data, linked to the spectrum through the multipole moments which are in turn defined through the two-point correlation function of the temperature fluctuations [281]. Let us note however, that such oscillations may be due not only to a bounce, but also to transplanckian effects [281] or to non-standard initial conditions in the framework of hybrid inflation [90].

We would like to close by pointing out that although they do not yet give a complete description of the universe, a better understanding of bouncing models in classical GR should be attempted since they are inevitably imposed upon us by the apparently observed violation of the strong energy condition. It must also be noted that there are at least two more reasons to attempt this task. First, the current solution to the problems of the standard cosmological models (namely inflation) is successful, but has several problems (see Section 1). Second, even if bouncing models do not succeed in yielding a complete description of the universe (thus offering an alternative to inflation¹²⁰), they may throw light upon the singularity problem (an issue in which inflation has nothing to say).

Summing up, we have seen in this review that bouncing universes have some attractive features, but they are not complete yet: much work is needed to achieve a stage in which their predictions can match those of the cosmological standard model. Therefore, we hope this review encourages further developments in nonsingular cosmologies.

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¹¹⁸ We have also seen that there are eternal (non-bouncing) universes, that start in a de Sitter regime.

¹¹⁹ Some bouncing models in GR were severely restricted in [389], using SNIa data, CMB analysis, nucleosynthesis, and the age of the oldest high-redshift objects.

¹²⁰ The comparison of bouncing models with the inflationary scenario has been undertaken in several articles (see for instance Refs. [179,263]).

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