

Closed Lightlike Curves in Non-linear Electrodynamics

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Abstract

We show that non-linear electrodynamics may induce a photon to follow a closed path in spacetime. We exhibit a specific case in which such closed lightlike curve (CLC) appears.

I. INTRODUCTION

A. Introductory Remarks

One of the most elegant characteristics that singles out the gravitational interaction consists in the possibility – turned into an actual formulation of gravity by Einstein – of associating gravitational phenomena to the metric structure of the spacetime. Such a geometrical view is impossible to be mimicked by processes involving other kinds of forces. The main reason for this is that all other known forces do not show the universal property that is typical of gravity. Indeed, this was the main drawback that induced Einstein unified program to its failure. However an interesting analogy with the Einstein way of looking into some processes appeared recently, carrying a less ambitious program but having a deep connection with it. It consists in a fresh method that – taking into account such limitation of non-universality – looks for special physical situations that allow an equivalent description in terms of an effective modification of the geometry of spacetime. In other words: even if a given kind of force cannot be geometrized, one can discover some special (and non-trivial) situations in which a restricted geometrization is possible. It is clearly understood that it is by no means a geometrization scheme in a broad sense but only a very limited one, although very useful, as we shall see. The interest on this arises, of course, from the possibility of applying such geometrization in a sufficiently large and meaningful set of examples. This is precisely the situation that one encounters in some interesting and diverse circumstances which has led to the

claim that nongravitational processes can indeed simulate modifications of the geometry of spacetime.

Just to consider two cases that have been presented in a coherent and self-contained way we refer to [1] and [2] that deal respectively with:

- The propagation of photons in non-linear electrodynamics;
- Processes involving certain properties of superfluid ³He.

In this paper we will deal only with the first case. Before going into this let us make another rather general comment to clarify our work here. We will be concerned with the propagation of photons in a nonlinear electrodynamics in terms of a modification of the metric of the spacetime. Such modification is nothing but an effective structure that yields an equivalent description of the photon paths. This should not be taken as an universal modification of the geometry of the spacetime. We shall see however that such geometric tool is very powerful and allows the analysis of light propagation to be accomplished in a very simple way. Besides, with this method we can transpose part of the behavior of photons from the well-known combined Maxwell-Einstein framework to the nonlinear case of electrodynamics. An example of this analogy will be presented in this paper. It concerns the possibility of the presence of photons closed paths in spacetime. We will see that the remarkable Gödel analysis of the existence of closed timelike curves (CTC) in a rotating universe can be transposed to the case of photons in non-linear electrodynamics. The electromagnetic field generated by a charged string yields the possibility of the existence of closed lightlike curves (CLC) for the photons.

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B. Synopsis

In the last years there has been an increasing interest on properties of the gravitational field that describe, within the context of General Relativity (GR), geometries that allow the appearance of closed timelike curves (CTC) (see, for instance [5], [6], [7], [8], [9]). Such unusual geometries, exact solutions of the classical Einstein equations of GR, pose a thorough problem of compatibility in the realm of field theory — just to quote one difficulty — and it is a real challenge to deal with them. As an example, we can point out the case of traversable wormholes that allow the existence of two nonequivalent paths for the possible travel of a real observer to go from one point P of spacetime to another point Q inducing thus the existence of a closed path in spacetime. Gödel cosmological solution is another case which also provides such kind of undesirable paths. The attraction for such geometries rests on the deep understanding of the theory allowed by their analysis. In the present paper we show that similar paths can be generated in configurations of pure electromagnetic field in a non-linear regime.

In order to achieve such a proof we must first review some recent papers that show this hidden geometrical character of the photon propagation in nonlinear electrodynamics [1], [3], [4]. We will recover such results in a very simple way in the next section.

C. Definitions and notations

We call the electromagnetic tensor $F_{\mu\nu}$, while its dual $F_{\mu\nu}^*$ is

$$F_{\alpha\beta}^* \doteq \frac{1}{2} \eta_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}, \quad (1)$$

where $\eta_{\alpha\beta\mu\nu}$ is the completely antisymmetric Levi-Civita tensor; the Minkowski metric tensor is represented by its standard form $\eta^{\mu\nu}$. The two invariants constructed with these tensors are defined as

$$F \doteq F^{\mu\nu} F_{\mu\nu}, \quad (2)$$

$$G \doteq F^{\mu\nu} F_{\mu\nu}^*. \quad (3)$$

Once the modifications of the linearity of electrodynamics which will be dealt here with do not break the gauge invariance of the theory, we can restrict our analysis to the general form of the modified Lagrangian for electrodynamics, written as a functional of the above invariants. In the present paper we limit our analysis to the case in which the Lagrangian depends only on F :

$$L = L(F). \quad (4)$$

We denote by L_F the derivative of the Lagrangian L with respect to the invariant F ; and similarly for the higher order derivatives. We are particularly interested in the derivation of the characteristic surfaces which guide the propagation of the field discontinuities.

II. THE METHOD OF THE EFFECTIVE GEOMETRY

We will make a short review of the Hadamard method in order to obtain the propagation equations for the discontinuities of the electromagnetic field. We limit our analysis to the case in which all modifications on the linear electrodynamics can be described by a Lagrangian L as in Eq. (4)

Let Σ be a surface of discontinuity for the electromagnetic field. Following Hadamard [1] we assume that the field itself is continuous when crossing Σ , while its first derivative presents a finite discontinuity. We accordingly set

$$[F_{\mu\nu}]_{\Sigma} = 0, \quad (5)$$

and

$$[\partial_{\lambda} F_{\mu\nu}]_{\Sigma} = f_{\mu\nu} k_{\lambda}, \quad (6)$$

in which the symbol

$$[J]_{\Sigma} \equiv \lim_{\delta \rightarrow 0^+} (J|_{\Sigma+\delta} - J|_{\Sigma-\delta})$$

represents the discontinuity of the arbitrary function J through the surface Σ characterized by the equation $\Sigma(x^{\mu}) = \text{constant}$. The tensor $f_{\mu\nu}$ is called the discontinuity of the field, and

$$k_{\lambda} = \partial_{\lambda} \Sigma \quad (7)$$

is the propagation vector.

The equations of motion are

$$\partial_{\nu} (L_F F^{\mu\nu}) = 0. \quad (8)$$

Following the definitions and procedure presented above one gets from the discontinuity of the equation of motion Eq. (8):

$$L_F f^{\mu\nu} k_{\nu} + 2L_{FF} \xi F^{\mu\nu} k_{\nu} = 0, \quad (9)$$

where ξ is defined by

$$\xi \equiv F^{\alpha\beta} f_{\alpha\beta}. \quad (10)$$

The cyclic identity yields

$$f_{\mu\nu} k_{\lambda} + f_{\nu\lambda} k_{\mu} + f_{\lambda\mu} k_{\nu} = 0. \quad (11)$$

Multiplying this equation by $k^{\lambda} F^{\mu\nu}$ gives

$$\xi k_{\nu} k_{\mu} \gamma^{\mu\nu} + 2 F^{\mu\nu} f_{\nu\lambda} k^{\lambda} k_{\mu} = 0, \quad (12)$$

in which $\gamma_{\mu\nu}$ is the Minkowski metric tensor written in an arbitrary coordinate system. From the Eq. (9) it results:

$$f_{\mu\nu} k^{\nu} = -2 \frac{L_{FF}}{L_F} \xi F_{\mu\nu} k^{\nu}. \quad (13)$$

After some algebraic manipulations the equation of propagation of the disturbances is obtained:

$$\{\gamma^{\mu\nu} + \Lambda^{\mu\nu}\} k_\mu k_\nu = 0 \quad (14)$$

in which the quantity $\Lambda^{\mu\nu}$ is

$$\Lambda^{\mu\nu} \equiv -4 \frac{L_{FF}}{L_F} F^{\mu\alpha} F_\alpha{}^\nu. \quad (15)$$

It then follows that the photon path is kinematically described by

$$g^{\mu\nu} k_\mu k_\nu = 0, \quad (16)$$

where the effective metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = L_F \gamma^{\mu\nu} - 4 L_{FF} F^\mu{}_\lambda F^{\lambda\nu}. \quad (17)$$

Furthermore, once the wave vector k_α is a gradient, the photon path is a geodesic in the effective geometry [1]. We re-obtained then the remarkable result that the discontinuities of the electromagnetic field in a nonlinear electrodynamics propagates along **null geodesics** of an effective geometry which depends on the properties of the background field¹.

From the general expression of the energy-momentum tensor for an electromagnetic theory $L = L(F)$ we have

$$T_{\mu\nu} = -4L_F F_\mu{}^\alpha F_{\alpha\nu} - L \gamma_{\mu\nu}. \quad (18)$$

We can then re-write the effective geometry in a more appealing form in terms of the energy momentum tensor. We obtain, using Eq. (18) into Eq. (17)

$$g^{\mu\nu} = \mathcal{M} \gamma^{\mu\nu} + \mathcal{N} T^{\mu\nu}, \quad (19)$$

where the functions \mathcal{M} and \mathcal{N} are given by

$$\mathcal{M} = L_F + \frac{L L_{FF}}{L_F}, \quad (20)$$

$$\mathcal{N} = \frac{L_{FF}}{L_F} \quad (21)$$

As a consequence of this, the Minkowskian norm of the propagation vector k_μ reads

$$\gamma^{\mu\nu} k_\mu k_\nu = -\frac{\mathcal{N}}{\mathcal{M}} T^{\mu\nu} k_\mu k_\nu. \quad (22)$$

¹The proof that the path is indeed a geodesics is given in the appendix B.

III. CHARGED STRING

The physical system we will analyse consists in a (idealized infinitely long) thin charged cylinder². The flat Minkowskian background geometry written in a (t, r, φ, z) coordinate system takes the form

$$ds^2 = dt^2 - dr^2 + 2h_0 d\varphi dt + g(r) d\varphi^2 - dz^2 \quad (23)$$

where $g(r) = h_0^2 - \omega^2 r^2$. Although locally such geometry is Minkowskian, it can be associated to a spinning string, due to its global properties. We shall see that this has no effect on our analysis, once CLC's do not exist in the Maxwell-Einstein theory but only in the case of nonlinear electrodynamics. This shows that our example of CLC is not a gravitational effect. Furthermore, one could set $\omega = 1$ in all calculations without any qualitative changing in our result. In order to avoid any artificial trouble with causality we will limit the range of the coordinate r to be strictly larger than r_0 , that is, $r > r_0$ where $r_0^2 = h_0^2/\omega^2$. In this domain of validity, this system is regular and well defined [11].

From the symmetry properties of this system, the unique non-null component of the electric field is $F_{01} = E(r)$. In this case³, the equation of motion reduces to

$$L_F E = \frac{Q}{r} \quad (24)$$

where Q is a constant. We are interested here in the analysis of the propagation of electromagnetic waves in such background. Following our previous treatment the photons propagate as if the metric structure of space-time were changed into an effective Riemannian geometry. From Eq.(17) we obtain the components of the effective metric. The non-vanishing covariant components⁴ are:

$$g_{tt} = \frac{\omega^2 r^2}{h_0^2 + \Psi \omega^2 r^2} \quad (25)$$

$$g_{t\varphi} = h_0 g_{tt} \quad (26)$$

$$g_{rr} = \frac{1}{\Lambda} \quad (27)$$

$$g_{\varphi\varphi} = -\Psi \omega^2 r^2 g_{tt} \quad (28)$$

²We neglect all gravitational effects once there is no substantial difference introduced by gravity, as far as the phenomenon we are interested to exhibit here is concerned.

³In order to avoid any difficulty with the coordinates we take the radius of the charged string to be greater than the value r_0 .

⁴See the appendix A.

$$g_{zz} = -1, \quad (29)$$

where Ψ and Λ are

$$\Psi = 1 - \left(\frac{h_0}{\omega r}\right)^2 - 4 \frac{L_{FF} E^2}{L_F} \quad (30)$$

$$\Lambda = -1 - 4 \frac{L_{FF} E^2}{L_F}. \quad (31)$$

The photon paths are null geodesics in such modified geometry. Let us consider the curve defined by the equations $t = \text{constant}$, $r = \text{constant}$, and $z = \text{constant}$. Along such a curve the element of length reduces to

$$ds_{eff}^2 = -\Psi \omega^4 r^4 \left(\frac{1}{h_0^2 + \Psi \omega^2 r^2} \right) d\varphi^2. \quad (32)$$

Thus, for the photon to follow such a path the radius $r = r_c$ must be such that $\Psi(r_c) = 0$.

Let us emphasize that the possibility of the presence of CLC's depends crucially on the non-linearity of the electromagnetic field. Indeed, it is a direct consequence of the form of the above geometry that Ψ can vanish only if L_{FF} is different from zero. In the linear Maxwell electrodynamics this phenomenon is forbidden⁵. Thus we are allowed to claim that it is a new property which depends on the non-linearity of the electromagnetic process. This is a general formalism valid for arbitrary form of the Lagrangian. We now turn to a specific example in which such situation occurs.

A. A toy model

We set for the nonlinear Lagrangian the form⁶:

$$L = \frac{b^2}{2} \left(\sqrt{1 - \frac{F}{b^2}} - 1 \right) \quad (33)$$

in which b is an arbitrary constant.

A solution of the equation for the electric field yields

$$E = 4bQ (b^2 r^2 - 32Q^2)^{-\frac{1}{2}}. \quad (34)$$

⁵Let us remark that a similar analysis on the photon path can be made for Maxwell theory in a non-linear dielectric medium. This is a direct consequence of the propagation equations in a non-linear dielectric medium as it was shown in [1].

⁶This form is very similar to Born-Infeld (BI) model. However, the sign in the field term inside the square-root is opposite to that used in BI Lagrangian. This makes a crucial difference in what concerns the appearance of CLC, as shown in the text.

The field must be defined for any value of r larger than r_0 . Thus, the minimum value of the radius r_{min} that follows from this expression must be larger than $\left(\frac{h_0}{\omega}\right)^2$, yielding a compromise between the constants. Using this value on the expression of the effective metric gives

$$ds_{eff}^2 = \frac{r^2}{r^2 - l^2} dt^2 + \frac{2h_0 r^2}{r^2 - l^2} dt d\varphi - \frac{r^2}{r^2 + l^2} dr^2 - \omega^2 r^2 \frac{r^2 - l^2 - \frac{h_0^2}{\omega^2}}{r^2 - l^2} d\varphi^2 - dz^2, \quad (35)$$

where $l^2 \equiv \frac{32Q^2}{b^2}$. The value for which Ψ vanishes is given by

$$r_c^2 = \left(\frac{h_0}{\omega}\right)^2 + \frac{32Q^2}{b^2}.$$

For $r = r_c$ the photon follows a closed spacetime path.

IV. FINAL COMMENTS

It has been known from more than half a century that gravitational processes allows the existence of closed paths in spacetime. This led to the belief that this strange situation occurs uniquely under the effect of gravity. In the present paper we have shown that this is not the case. Indeed, we show here that photons can follow closed paths (CLC) due to electromagnetic forces in a non-linear regime. We presented an specific example of a theory in which CLC exists. In the limit case where the non-linearities are neglected the presence of CLC is no more possible. Thus we are allowed to claim that this new property depends crucially on the non-linearity of the electromagnetic process and it is not possible to exist in Maxwell theory.

This shows that the existence of CLC is **not** an exclusive property of gravitational interaction: it can exists also in pure electromagnetic processes depending on the non-linearities of the background field. The existence of such CLC in both gravitational and electromagnetic processes asks for a deep review of the causal structure displayed by the photon path.

V. APPENDIX A: THE INVERSE METRIC

In the case in which the Lagrangian depends only on the invariant F the effective geometry takes the form

$$g^{\mu\nu} = L_F \gamma^{\mu\nu} - 4 L_{FF} F^\mu{}_\lambda F^{\lambda\nu} \quad (36)$$

The inverse metric, the covariant tensor $g_{\mu\nu}$ defined by

$$g^{\mu\nu} g_{\nu\alpha} = \delta_\alpha^\mu$$

can be easily evaluated by taking into account the identities

$$F^\mu{}_\lambda F^{\lambda\nu} - F^{*\mu}{}_\lambda F^{*\lambda\nu} = -\frac{1}{2} F \gamma^{\mu\nu},$$

and

$$F^{*\mu}{}_\lambda F^{\lambda\nu} = -\frac{G}{4} \gamma^{\mu\nu}.$$

A direct manipulation yields the result

$$g_{\mu\nu} = A \gamma_{\mu\nu} + B F_\mu{}^\lambda F_{\lambda\nu} \quad (37)$$

where A and B are

$$A = \frac{1}{R} (L_F + 2 F L_{FF}),$$

$$B = \frac{4 L_{FF}}{R},$$

and R is defined by

$$R = L_F^2 + L_{FF} (2 F L_F - G^2).$$

VI. APPENDIX B: THE EFFECTIVE NULL GEODESICS

The geometrical relevance of the effective geometry (17) goes beyond its immediate definition. Indeed, we will demonstrate here that the integral curves of the vector k_ν (*i.e.*, the trajectories of such nonlinear *photons*) are in fact geodesics. In order to achieve this result it will be required an underlying Riemannian structure for the manifold associated with the effective geometry. In other words, this means a set of Levi-Civita connection coefficients $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$, by means of which there exists a covariant differential operator ∇_λ (the *covariant derivative*) such that

$$\nabla_\lambda g^{\mu\nu} \equiv g^{\mu\nu}{}_{;\lambda} \equiv g^{\mu\nu}{}_{,\lambda} + \Gamma^\mu{}_{\sigma\lambda} g^{\sigma\nu} + \Gamma^\nu{}_{\sigma\lambda} g^{\sigma\mu} = 0. \quad (38)$$

From (38) it follows that the effective connection coefficients are completely determined from the effective geometry by the usual Christoffel formula.

Contracting (38) with $k_\mu k_\nu$ results

$$k_\mu k_\nu g^{\mu\nu}{}_{,\lambda} = -2 k_\mu k_\nu \Gamma^\mu{}_{\sigma\lambda} g^{\sigma\nu}. \quad (39)$$

Differentiating (16) and remembering $g^{\mu\nu} = g^{\nu\mu}$ one gets

$$2 k_{\mu,\lambda} k_\nu g^{\mu\nu} + k_\mu k_\nu g^{\mu\nu}{}_{,\lambda} = 0. \quad (40)$$

Inserting (39) for the last term on the left hand side of (40) we obtain

$$g^{\mu\nu} k_{\mu,\lambda} k_\nu - g^{\sigma\nu} \Gamma^\mu{}_{\sigma\lambda} k_\mu k_\nu = 0. \quad (41)$$

Relabeling contracted indices we can rewrite (41) as

$$g^{\mu\nu} k_{\mu;\lambda} k_\nu \equiv g^{\mu\nu} [k_{\mu,\lambda} - \Gamma^\sigma{}_{\mu\lambda} k_\sigma] k_\nu = 0. \quad (42)$$

Now, as the propagation vector $k_\mu = \Sigma_{,\mu}$ is an exact gradient one can write $k_{\mu;\lambda} = k_{\lambda;\mu}$. With this identity and defining $k^\nu \equiv g^{\mu\nu} k_\mu$ equation (42) reads

$$k_{\mu;\lambda} k^\lambda = 0, \quad (43)$$

which states that k_μ is a geodesic vector. Remembering it is also a null vector (with respect to the effective geometry $g^{\mu\nu}$) its integral curves are thus null geodesics. We can restate our previous demonstration as:

The discontinuities of the electromagnetic field in a nonlinear electrodynamics propagates along **null geodesics** of an effective geometry which depends on the properties of the background field.

Thus, photons follow geodesics in an effective geometry. Since the photon does not have electric charge, the force that acts on it in a non-linear electromagnetic field has a distinct character than that of the Lorentz force. Indeed, from the geodesic equation, the electromagnetic field acts on the photon by means of a force given by

$$f^\alpha = -\Delta^\alpha{}_{\mu\nu} k^\mu k^\nu \quad (44)$$

in which the quantity $\Delta^\alpha{}_{\mu\nu}$ can be displayed in terms of the effective geometry $g^{\mu\nu}$ and its inverse $g_{\mu\nu}$ given above by the Christoffel form.

$$\Delta^\alpha{}_{\mu\nu} = \frac{1}{2} (L_F \gamma^{\alpha\beta} + \Phi^{\alpha\beta}) (\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu})$$

where $\Phi^{\alpha\beta} \equiv -4 L_{FF} F^\alpha{}_\lambda F^{\lambda\beta}$. Hence, it follows that the net effect of the force that the field exerts on the photon has very similar properties as the gravitational force. This is precisely the reason that allows us to interpret the action of the electromagnetic field on the photon to be a mimic of the behavior of a massless particle in a gravitational field.

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