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2000 Class. Quantum Grav. 17 3821

(http://iopscience.iop.org/0264-9381/17/18/316)

Class. Quantum Grav. **17** (2000) 3821–3831. Printed in the UK PII: S0264-9381(00)50629-7

Singularities in general relativity coupled to nonlinear electrodynamics

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Received 3 May 2000

Abstract. We study here some consequences of the nonlinearities of the electromagnetic field acting as a source of Einstein's equations on the propagation of photons. We restrict to the particular case of a 'regular black hole', and show that there exist singularities in the effective geometry. These singularities may be hidden behind a horizon or be naked, according to the value of a parameter. Some unusual properties of this solution are also analysed.

PACS numbers: 0420D, 0420J

1. Introduction

It is a well known fact that some of the most important solutions of Einstein's field equations (e.g. Friedmann–Robertson–Walker and Schwarzschild) are singular. However, our understanding of the nature of these singularities is still incomplete. For instance, the cosmic censorship conjecture was put forward by Penrose in 1969 [1], but there is still no general proof of it. As a consequence of this lack of understanding, solutions that are everywhere regular and share some of the properties of singular solutions deserve attention. This is precisely the case of the 'regular black hole' spacetimes recently exhibited in [2–4]. These solutions were obtained for a very special type of source: an electric field that obeys a nonlinear electrodynamics. The authors of [2] analysed some of the features of the solution, but left aside others that are relevant. We shall re-examine this solution in detail. More importantly, we shall show in this particular example the far-reaching consequences of the fact that in nonlinear electromagnetism photons *do not* propagate along null geodesics of the background geometry. They propagate instead along null geodesics of an effective geometry, which depends on the nonlinearities of the theory. This result, derived by Plebański for Born–Infeld electrodynamics [5], was generalized for any nonlinear theory by Gutiérrez *et al* [6], and later independently rediscovered by Novello *et al* [7]. Let us mention that the propagation of photons beyond Maxwell electrodynamics has been studied in several different situations. It has been investigated in curved spacetime, as a consequence of non-minimal coupling of electrodynamics with gravity [8–10], and in non-trivial QED vacua as an effective modification induced by quantum fluctuations [11–13]. Nearly always, these analysis have had some unexpected results. As an example, let us mention the possibility of *faster and slower-than-light* photons [11].

Our main concern in this paper will be then to show that one must consider the modifications on the trajectories of the photons induced by the nonlinearities of the

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electromagnetic theory in order to give a complete characterization of spacetimes with a nonlinear electromagnetic source. The structure of the paper is the following. A summary of the solution given in [2] and the properties studied there will be given in section 2, along with some interesting properties that went unnoticed before. In section 3 we briefly review the origin of the effective geometry for photons in nonlinear electrodynamics. In section 4 we shall use the method of the effective geometry to study the features of the structure that photons see when travelling in the geometry given in [2]. We close with some conclusions.

2. Details of the solution

Ayón-Beato and García [2] have found an exact solution of Einstein's equations in the presence of a nonlinear electromagnetic source. The relevant equations are derived from the action [14]

$$\mathcal{S} = \int \mathrm{d}^4 x \left[\frac{1}{16\pi} R - \frac{1}{4\pi} \mathcal{L}(F) \right], \tag{1}$$

where R is the curvature scalar and $\mathcal{L}$ is a nonlinear function of $F \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. Following [2, 5] this system could also be described using another function obtained by means of a Legendre transformation:

$$\mathcal{H} \equiv 2F\mathcal{L}_F - \mathcal{L}. \tag{2}$$

($\mathcal{L}_F$ denotes the derivative of $\mathcal{L}$ with respect to F.) With the definition

$$P_{\mu\nu} \equiv \mathcal{L}_F F_{\mu\nu}, \tag{3}$$

it can be shown that $\mathcal{H}$ is a function of $P \equiv \frac{1}{4} P_{\mu\nu} P^{\mu\nu} = (\mathcal{L}_F)^2 F$, i.e. $\mathrm{d}\mathcal{H} = (\mathcal{L}_F)^{-1} \mathrm{d}((\mathcal{L}_F)^2 F) = \mathcal{H}_P \, \mathrm{d}P$. With the help of $\mathcal{H}$ one could express the nonlinear electromagnetic Lagrangian in the action (1) as $\mathcal{L} = 2P\mathcal{H}_P - \mathcal{H}$, which depends on the antisymmetric tensor $P_{\mu\nu}$. The solution of Einstein's equations coupled to nonlinear electrodynamics obtained in [2] was derived from the following source:

$$\mathcal{H}(P) = P \frac{\left(1 - 3\sqrt{-2q^2 P}\right)}{\left(1 + \sqrt{-2q^2 P}\right)^3} - \frac{3}{2q^2 s} \left(\frac{\sqrt{-2q^2 P}}{1 + \sqrt{-2q^2 P}} \right)^{5/2}, \tag{4}$$

where $s = |q|/2m$ and the invariant P is a negative quantity. The corresponding Lagrangian is given by

$$\mathcal{L} = P \frac{\left(1 - 8\sqrt{-2q^2 P} - 6q^2 P\right)}{\left(1 + \sqrt{-2q^2 P}\right)^4} - \frac{3}{4q^2 s} \frac{(-2q^2 P)^{5/4} \left(3 - 2\sqrt{-2q^2 P}\right)}{\left(1 + \sqrt{-2q^2 P}\right)^{7/2}}. \tag{5}$$

From equation (1) we obtain the following equations of motion:

$$G_\mu{}^\nu = 2(\mathcal{H}_P P_{\mu\lambda} P^{\nu\lambda} - \delta_\mu{}^\nu (2P\mathcal{H}_P - \mathcal{H})), \tag{6}$$

$$\nabla_\mu P^{\alpha\mu} = 0. \tag{7}$$

This system was solved in [2], and the explicit form of the solution is the following:

$$\begin{aligned} \mathrm{d}s^2 = {} & \left[1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2 r^2}{(r^2 + q^2)^2} \right] \mathrm{d}t^2 \\ & - \left[1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2 r^2}{(r^2 + q^2)^2} \right]^{-1} \mathrm{d}r^2 - r^2 \, \mathrm{d}\Omega^2, \end{aligned} \tag{8}$$

$$E_r = q r^4 \left[\frac{r^2 - 5q^2}{(r^2 + q^2)^4} + \frac{15}{2} \frac{m}{(r^2 + q^2)^{7/2}} \right]. \tag{9}$$

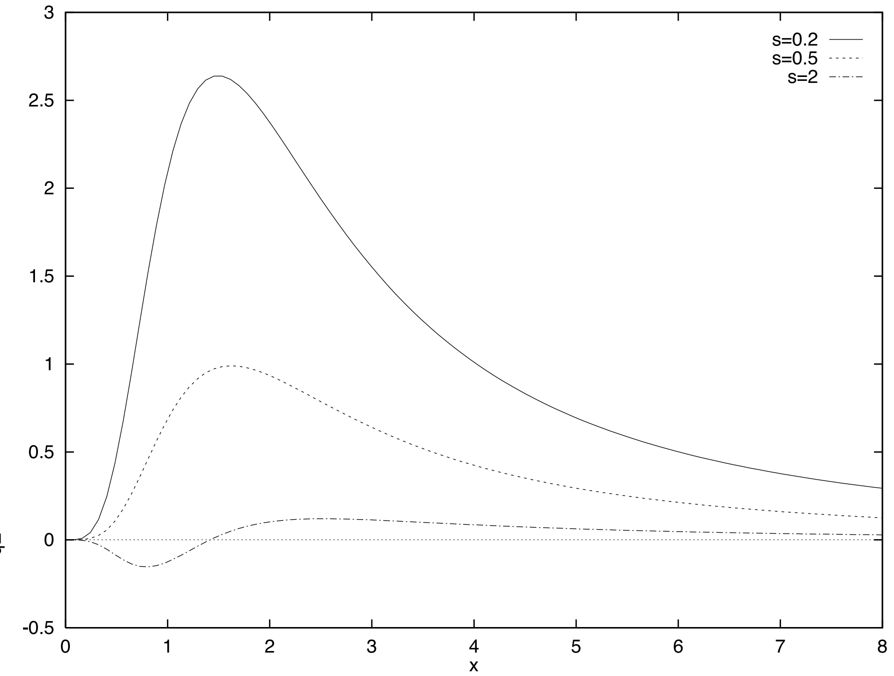

Figure 1. Electric field times the electric charge q as a function of x for different values of s.

By means of the substitution $x = r/|q|$ we can rewrite g_{tt} and E_r as follows:

$$g_{tt} = A(x, s) \equiv 1 - \frac{1}{s}\frac{x^2}{(1+x^2)^{3/2}} + \frac{x^2}{(1+x^2)^2} \tag{10}$$

$$E_r = \frac{x^4}{q}\left[\frac{x^2-5}{(x^2+1)^4} + \frac{15}{4s}\frac{1}{(x^2+1)^{7/2}}\right]. \tag{11}$$

The result of the analysis made in [2] is that this metric describes a regular black hole. The position of the horizons was identified there with the values of the coordinate x for which g_{tt} is zero. These are given by

$$s = \frac{x^2\sqrt{x^2+1}}{x^4+3x^2+1}. \tag{12}$$

Accordingly, the solution has two horizons (for $0 < s < 0.317$), one horizon (for $s = 0.317$), or no horizons (for $s > 0.317$). It was also stated that this solution is regular, on the basis of the finiteness of the three invariants R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$†.

Let us point out now some features of the solution described by equations (10) and (11) that were not noticed in [2]. First, the behaviour of the radial component of the electric field depends on the value of s. Specifically, E_r may have a zero; its position is given by

$$s = -\frac{15}{4}\frac{\sqrt{x^2+1}}{x^2-5}. \tag{13}$$

Consequently, E_r does not have zeros for $0 < s < \frac{3}{4}$. For $s \geqslant \frac{3}{4}$, E_r has one zero located in the interval $(0, \sqrt{5})$ of the coordinate x. These features of the electric field are depicted in figure 1‡.

† We have checked that all the components of R_{ABCD} and C_{ABCD} with respect to a static observer are finite at $r = 0$.
‡ The plots in this paper have been made with gnuplot [15].

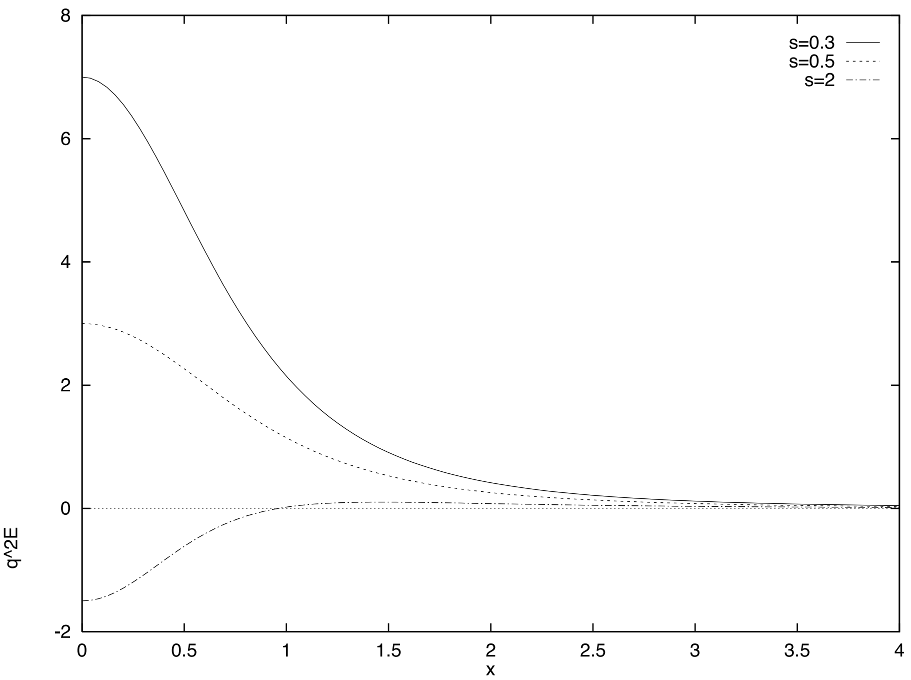

Figure 2. Energy density of the electric field times q^2 as a function of x for different values of s.

Another salient feature of E_r is that its energy density, calculated as the G_t^t component of the Einstein tensor†, may be negative for some interval of x. In fact, the expression

$$G_t^t = \rho = \frac{1}{sq^2}\frac{s\sqrt{1+x^2}\,(x^2-3)+3(x^2+1)}{(1+x^2)^{7/2}} \tag{14}$$

is zero for

$$s = -3\frac{\sqrt{1+x^2}}{x^2-3}. \tag{15}$$

For $s < 1$, the energy is always positive, but for $s \geqslant 1$ it has a zero given by equation (15). Figure 2 illustrates the situation.

3. Effective geometry for photons

In this section we give a summary of the method of the effective geometry [7]. We will deal here only with the case in which the Lagrangian of the nonlinear electromagnetic theory is a function of F only. The general case in which $\mathcal{L}$ also depends on $G = \frac{1}{2}F^{\mu\nu}\eta^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}$ is analysed in [7]. Based on the framework introduced by Hadamard [17], Novello *et al* showed that the discontinuities of the electromagnetic field propagate according to the equation

$$(\mathcal{L}_F\eta^{\mu\nu} - 4\mathcal{L}_{FF}F^{\mu\alpha}F_\alpha{}^\nu)\,k_\mu k_\nu = 0, \tag{16}$$

† This and other calculations in this paper were done with the package *Riemann* [16].

where $\eta_{\mu\nu}$ is the (flat) background metric and k^μ is the propagation vector. This expression suggests that the self-interaction of the field $F^{\mu\nu}$ can be interpreted as a modification on the spacetime metric $\eta_{\mu\nu}$, leading to the effective geometry

$$g^{\mu\nu}_{(\mathrm{eff})} = \mathcal{L}_F \eta^{\mu\nu} - 4\mathcal{L}_{FF} F^{\mu}{}_{\alpha} F^{\alpha\nu}. \tag{17}$$

Note that only in the particular case of linear Maxwell electrodynamics the discontinuities of the electromagnetic field propagate along the null cones of the Minkowskian background.

The general expression of the effective geometry can be equivalently written in terms of the energy–momentum tensor, given by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta\Gamma}{\delta\gamma^{\mu\nu}}, \tag{18}$$

where Γ is the effective action

$$\Gamma \doteq \int \mathrm{d}^4x \sqrt{-\gamma} L, \tag{19}$$

and $\gamma_{\mu\nu}$ is the Minkowski metric written in an arbitrary coordinate system; γ is the corresponding determinant. In the case of one-parameter Lagrangians, $\mathcal{L} = \mathcal{L}(F)$, we obtain

$$T_{\mu\nu} = -4\mathcal{L}_F F_{\mu}{}^{\alpha} F_{\alpha\nu} - \mathcal{L}\eta_{\mu\nu}, \tag{20}$$

where we have chosen a Cartesian coordinate system in which $\gamma_{\mu\nu}$ reduces to $\eta_{\mu\nu}$. In terms of this tensor the effective geometry (17) can be rewritten as

$$g^{\mu\nu}_{(\mathrm{eff})} = \left(\mathcal{L}_F + \frac{\mathcal{L}\mathcal{L}_{FF}}{\mathcal{L}_F}\right)\eta^{\mu\nu} + \frac{\mathcal{L}_{FF}}{\mathcal{L}_F} T^{\mu\nu}. \tag{21}$$

It is shown in [7] that the field discontinuities propagate along the null geodesics of the effective geometry given by equation (21). This equation explicitly shows that the stress–energy distribution of the field is responsible for the deviation of the geometry felt by photons from its Minkowskian form†.

We will now show that the modification of the underlying spacetime geometry seen by photons due to nonlinear electrodynamics can also be described as if photons governed by Maxwell electrodynamics were propagating inside a dielectric medium. In this case, the electromagnetic field is represented by two antisymmetric tensors, the electromagnetic field $F_{\mu\nu}$ and the polarization field $P_{\mu\nu}$. For electrostatic fields inside isotropic dielectrics it follows that $P_{\mu\nu}$ and $F_{\mu\nu}$ are related by

$$P_{\mu\nu} = \epsilon(E) F_{\mu\nu} \tag{22}$$

where ϵ is the electric susceptibility. Comparing with equation (3) we see that we can make the identification

$$\mathcal{L}_F \longrightarrow \epsilon, \tag{23}$$

which implies

$$\mathcal{L}_{FF} \longrightarrow -\frac{\epsilon'}{4E}, \tag{24}$$

in which $\epsilon' \equiv \mathrm{d}\epsilon/\mathrm{d}E$ and $E^2 \equiv -E_\alpha E^\alpha > 0$. Therefore, every Lagrangian $\mathcal{L} = \mathcal{L}(F)$ which describes a nonlinear electromagnetic theory may be used as a convenient description of Maxwell theory inside isotropic nonlinear dielectric media. Conversely, results obtained

† For $T_{\mu\nu} = 0$, the conformal modification in (21) clearly leaves the photon paths unchanged.

in the latter context can be restated in terms of Lagrangians of nonlinear theories. Using this equivalence, the effective geometry can be rewritten as

$$g^{\mu\nu}_{(\mathrm{eff})} = \epsilon \eta^{\mu\nu} - \frac{\epsilon'}{E}\left(E^{\mu}E^{\nu} - E^{2}\delta^{\mu}_{t}\delta^{\nu}_{t}\right). \tag{25}$$

In other words,

$$g^{tt}_{(\mathrm{eff})} = \epsilon + \epsilon' E, \tag{26}$$

$$g^{ij}_{(\mathrm{eff})} = -\epsilon\delta^{ij} - \frac{\epsilon'}{E}E^{i}E^{j}. \tag{27}$$

This shows that the discontinuities of the electromagnetic field inside a nonlinear dielectric medium propagate along null cones of an effective geometry (given by equation (25)) which depends on the characteristics of the medium.

Although in [7] the background was flat, the method can also be used in a curved background. The reason is that the equations given in [7] are valid locally in any curved spacetime. Then from the equivalence principle it follows that the only change in equation (17) is that of $\eta_{\mu\nu}$ by $g_{\mu\nu}$.

4. Analysis of the 'regular black hole'

Using equations (26) and (27) it follows that the effective metric associated with a spherically symmetric solution of Einstein's equations is given by

$$\mathrm{d}s^2 = \frac{1}{\Phi(r)}\left[A(r)\,\mathrm{d}t^2 - A(r)^{-1}\,\mathrm{d}r^2\right] - \frac{r^2}{\mathcal{L}_F}\,\mathrm{d}\Omega^2, \tag{28}$$

where

$$\Phi = \epsilon + \frac{\mathrm{d}\epsilon}{\mathrm{d}E_r}E_r = -\frac{2q}{r^3}\frac{1}{\mathrm{d}E_r/\mathrm{d}r} \tag{29}$$

and

$$\epsilon = \frac{1}{E_r}\sqrt{\frac{-P_{\mu\nu}P^{\mu\nu}}{2}}. \tag{30}$$

For the case dealt with in the previous section, the function Φ takes the form

$$\Phi(x,s) = \frac{8(x^2+1)^5 s}{x^6(8x^4 s - 104 s x^2 + 80 s + 45x^2\sqrt{x^2+1} - 60\sqrt{x^2+1})}. \tag{31}$$

From equation (28) we see that the tt coefficient of the effective metric is given by the quotient $g^{(\mathrm{eff})}_{tt} = A/\Phi$. The function Φ^{-1} has real zeros for

$$s = -\frac{15}{8}\frac{\sqrt{x^2+1}\,(3x^2-4)}{x^4 - 13x^2 + 10}. \tag{32}$$

Taking into account that s must be positive, we conclude from equation (32) that the function Φ^{-1} has one zero for $s < \frac{3}{4}$ and two zeros for $s \geqslant \frac{3}{4}$. In both cases the zeros are in the interval $(0, 3.49)$ of the coordinate x.

It was shown in [2] that the metric coefficient g_{tt} given by equation (10) has two zeros for $s < 0.317$, one zero for $s = 0.317$ and no zeros for $s > 0.317$. The zeros in g_{tt} were identified in [2] with horizons. We see that due to the effective metric, the geometry seen by the photons is more complex than the geometry seen by ordinary matter. Taking into account

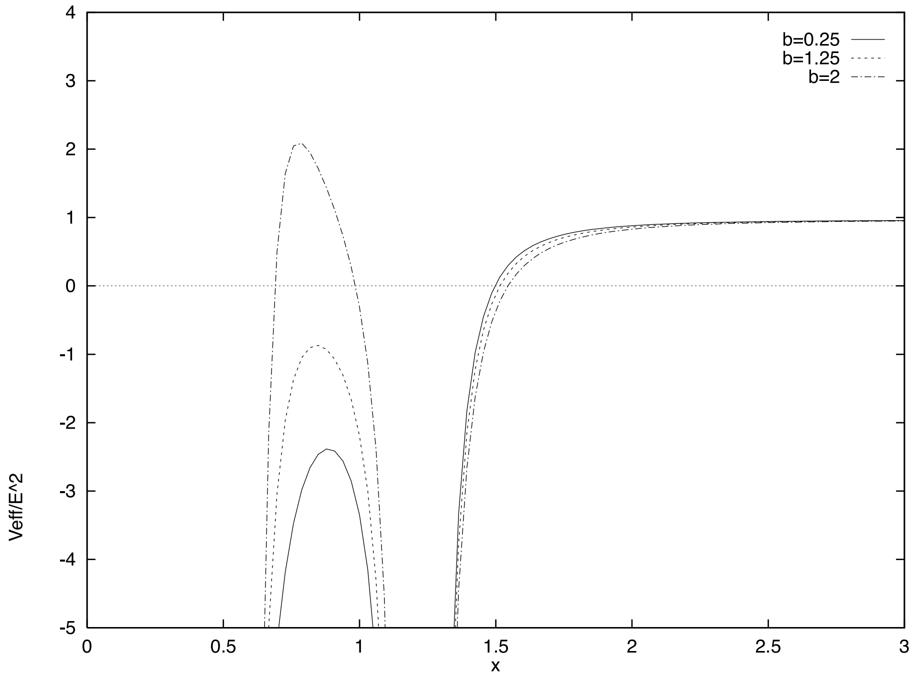

Figure 3. Effective potential V_{eff}/E_0^2 for $s = 0.2$ and different values of the impact parameter $b = h_0/E_0$.

the zeros of A and those of Φ^{-1} we conclude that $g_{tt}^{(\text{eff})}$ has three zeros for $s < 0.371$, two zeros for $s = 0.371$, one zero for $0.317 < s < \frac{3}{4}$, and again two zeros for $s \geqslant \frac{3}{4}$.

To determine the nature of the new zeros in the metric, it is useful to study the effective potential that is felt by the photons. The symmetries of the metric imply that there are two Killing vectors and consequently, two conserved quantities:

$$E_0 = g_{tt}\dot{t} \qquad \text{and} \qquad h_0 = \frac{r^2}{\mathcal{L}_F}\dot{\phi} \tag{33}$$

(the overdot denotes a derivative with respect to the affine parameter). Standard calculations (see, for instance, [18]) using $g_{\mu\nu}^{(\text{eff})}$ show that the effective potential for photons is given by

$$V_{\text{eff}} = (1 - \Phi^2)\frac{E_0^2}{2} + \frac{h_0^2}{x^2}\mathcal{L}_F A\Phi. \tag{34}$$

The explicit form of the effective potential is too involved to be displayed here. However, we note that V_{eff} has poles. One of them is at $x = 0$, and the others are given by the expression of the poles of Φ (see equation (32)), and those of $\mathcal{L}_F$ which are given by equation (13). $\mathcal{L}_F$ has no poles for $0 < s < \frac{3}{4}$, and one pole for $s \geqslant \frac{3}{4}$. Leaving aside the pole at $x = 0$, it follows that for $s < \frac{3}{4}$, the effective potential has only one pole, and for $s \geqslant \frac{3}{4}$, it has three poles. Those that originate in the singularities of the function Φ are in agreement with the extrema of the electric field, as shown by equation (31). We give in figures 3–5 plots of V_{eff} for different values of the relevant parameters.

Several comments are in order. The singularities in the potential suggest that the effective geometry itself may be singular. This is confirmed by the expression of the scalar curvature $R^{(\text{eff})}$, which diverges in the values of x given by equations (13) and (32). Let us analyse the

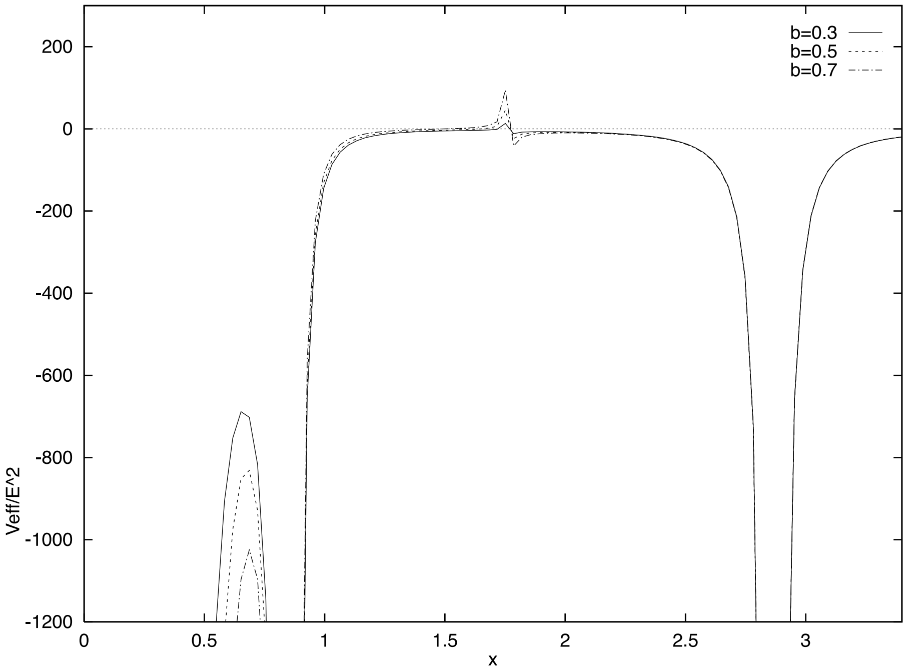

Figure 4. Effective potential V_{eff}/E_0^2 for $s = 4$ and different values of b. The interval $1.7 \leqslant x \leqslant 1.8$ is shown in detail in figure 5.

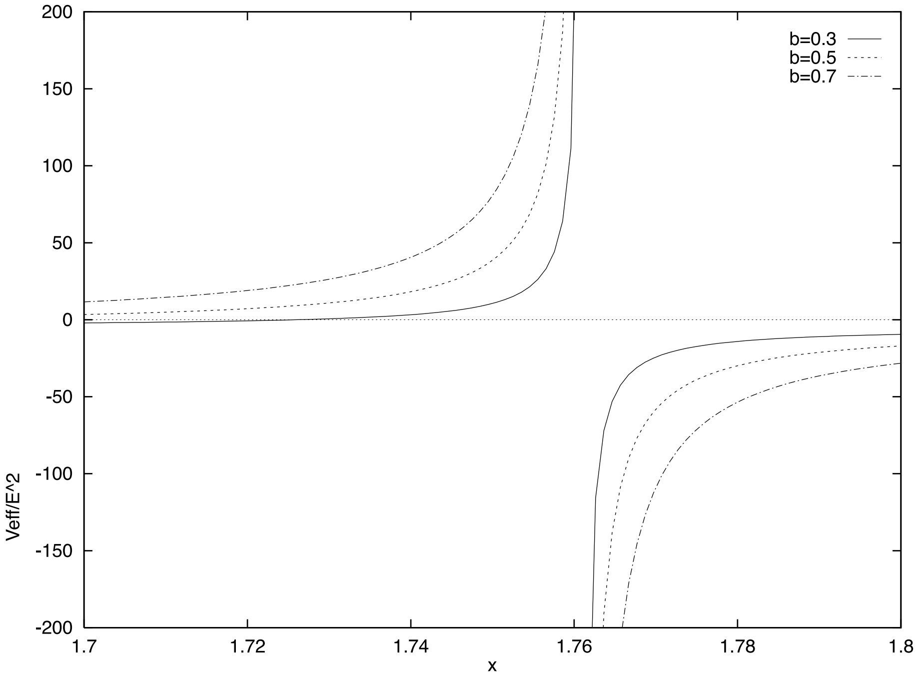

Figure 5. Effective potential V_{eff}/E_0^2 for $s = 4$ and different values of b. The singularity seen in this plot comes from the pole of $\mathcal{L}_F$.

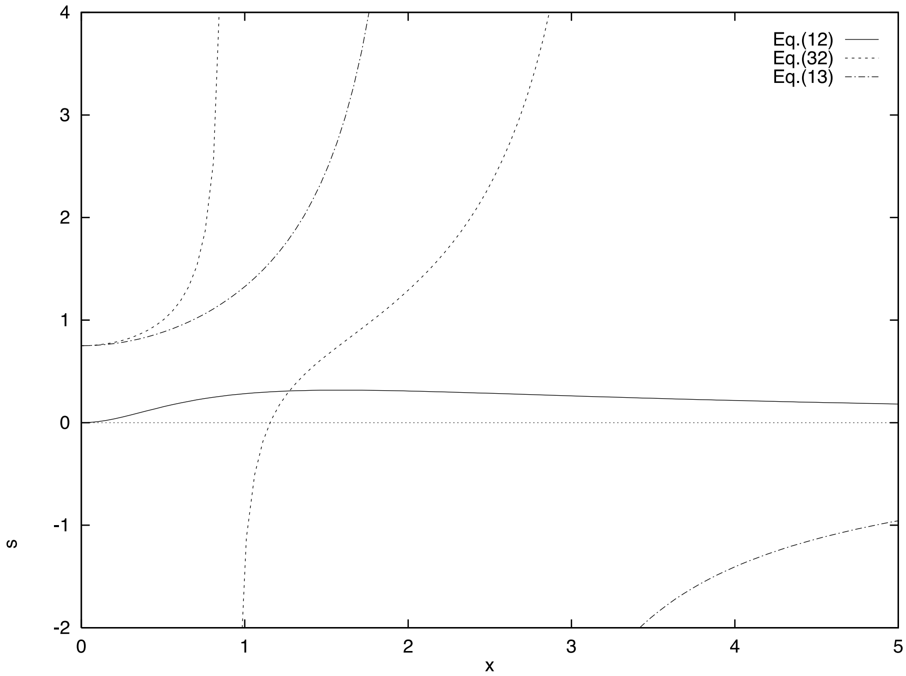

Figure 6. Positions of the zeros of the metric and spacetime singularities for a given s.

relative positions of these singularities felt by the photons and those of the metric coefficient $g_{tt}(x, s)$, given by equation (12). The information is conveniently summarized in figure 6.

We see that for a fixed $s \leqslant 0.317$ the singularities are situated inside the first horizon. However, for $s > 0.317$ the singularities are no longer hidden behind a horizon: we are then in the presence of naked singularities. We must remark that these singularities are only felt by photons. The rest of the matter follows the geodesics of the regular spacetime given in [2].

It can also be seen from the plot that for $s < 0.371$ the coordinate distance between the two horizons decreases for increasing s, up to $s = 0.371$, where the two horizons coalesce.

Before analysing the path of a photon coming from infinity, let us remark that there is a low potential barrier extending to the right of the outermost singularity for any value of the parameters. This barrier can be seen in figure 3, and it is also present to the right of figure 4. A low-energy photon incident from the right will then find this barrier, and will be deflected back to large values of x. This deflection will be more pronounced with increasing energy. When the energy of the photon is approximately that of the height of the barrier, the photon can orbit around the centre of the field in an unstable orbit. Finally, an incident photon with energy greater than the height of the barrier will inevitably encounter the first singularity.

It is easily seen from equation (34) that the potential goes to zero for large values of x. We have also analysed the effective potential for the case of a negative q, but the only quantitatively different result is a small increment of the innermost local maximum seen in figures 3 and 4.

We now move to another peculiar feature of the effective geometry. It is known that the effective potential for the Schwarzschild and Reissner–Nordström geometries is null in the case of photons with $h_0 = 0$. However, from equation (34) we see that in this case V_{eff} for the effective geometry reduces to

$$V_{\text{eff}} = (1 - \Phi^2)E_0^2. \qquad (35)$$

The dependence of this potential on Φ is the same as in equation (34), so the behaviour of V_{eff} with x in this case is qualitatively depicted in figures 3 and 4.

Let us finally point out some unusual geometrical properties of the metric seen by the photons. The effective metric has the same symmetries as the original metric given by equation (8). It can be easily shown that the time Killing vector $\partial/\partial t$ is null on the hypersurfaces determined by the zeros of $g_{tt}^{(\text{eff})}$.

Another interesting property of these surfaces is associated with the redshift of the photons. The redshift z of a source as measured by an observer with velocity u^μ can be defined in terms of the frequency by

$$1 + z = \frac{(u^\mu k_\mu)_{\text{emitter}}}{(u^\mu k_\mu)_{\text{observer}}}. \tag{36}$$

Considering a static observer for which $u^\mu = \delta_0^\mu/\sqrt{g_{tt}}$ this expression can be written as

$$1 + z = \left[\frac{\sqrt{g_{tt}}}{g_{tt}^{(\text{eff})}}\right]_{\text{em}} \left[\frac{g_{tt}^{(\text{eff})}}{\sqrt{g_{tt}}}\right]_{\text{obs}}. \tag{37}$$

Using the expression of the effective metric, and if the observer is at infinity,

$$1 + z = \frac{\Phi}{\sqrt{A}}. \tag{38}$$

We conclude then that the redshift diverges in two cases: when A is zero, and when Φ diverges (see figure 6).

5. Conclusion

The remarkable fact that in nonlinear electrodynamics the trajectories of photons are modified by the nonlinearities of the field equations has not been addressed frequently in the literature. The photons do not propagate following the null cones of the background metric but those of the effective metric. We have shown here the dramatic consequences that this has in a so-called regular black hole. In this case, there are singularities that are seen only by the photons. These singularities can either be hidden behind a horizon or naked, according to the value of the ratio $q/2m$. Let us remark that the existence of singularities in these types of solutions is a direct consequence of the existence of extrema of the electric field, as equation (29) shows. This is a general property which will always be present in any static and spherically symmetric solution of the system of equations (6) and (7) when the electromagnetic theory is nonlinear.

We have also shown that the effective potential to the right of the outermost singularity resembles that of Schwarzschild and Reissner–Nordström. However, contrary to what happens in Maxwell theory, photons with zero angular momentum travel under the influence of an effective potential that is different from zero.

We have also exhibited some unusual properties of the solution found in [2]. The electric field may have one or two extrema depending on the value of s. In the second case, it has a zero. Also, for certain values of s the energy of the electric field is negative in some coordinate range. There are at least two more properties, geometrical in origin, that are worthy of note. First, the time Killing vector of the effective geometry is null in the surfaces where the function Φ diverges. Second, the redshift measured by an observer far from the source diverges on the same surfaces. It is important to remark that these geometrical properties will be present in every solution with the same symmetries if the electric field has extrema.

To close, we would like to emphasize that ordinary matter follows geodesics of the background metric. However, the modifications of the metric induced by the nonlinearities of the electromagnetic field must always be taken into account when studying the propagation of photons. The above-mentioned properties are nothing but a consequence of the nonlinearities of the electromagnetic theory.

Acknowledgments

SEPB would like to acknowledge financial support from CONICET-Argentina. MN and JMS acknowledge financial support from CNPq. The authors would like to thank K Bronnikov for a useful remark.

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