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RESEARCH ARTICLE

Dragged metrics

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Abstract We show that the path of any accelerated body in an arbitrary spacetime geometry $g_{\mu\nu}$ can be described as a geodesic in a dragged metric $\hat{q}_{\mu\nu}$ that depends only on the background metric and on the motion of the body. Such procedure allows the interpretation of all kinds of non-gravitational force as modifications of the spacetime metric. This method of effective elimination of the forces by changing the metric of the substratum can be understood as a generalization of the d'Alembert principle applied to all relativistic processes.

Keywords Accelerated motion · Geodesics

1 Introduction

Since 1915 we know that the effect of the gravitational force on a body $\mathcal{A}$ can be geometrized [1]. This means that an accelerated path in Newtonian gravity can be assimilated by a geodesic in a curved geometry. This was possible due to the universality of the gravitational field and to the fact that all bodies acquire the same acceleration in a given gravitational field. What about other forces? Could it be possible to geometrize the effects of non-universal forces whose effects on different bodies are not the same? Indeed, in this paper, we show that

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- Following similarly the treatment for moving bodies as used in general relativity we show that any accelerated curve in Minkowski background can be mapped in a geodesic curve in another geometry $\hat{q}_{\mu\nu}$.
- The metric $\hat{q}_{\mu\nu}$ depends on the velocity and on the acceleration of the particle.

A concrete example of such procedure was presented by Gordon [2]. In his paper, he showed that the electromagnetic waves in a moving dielectric propagate as geodesics not in the background geometry $\eta_{\mu\nu}$ but instead in a metric

$$g^{\mu\nu} = \eta^{\mu\nu} + (\epsilon\mu - 1)\, v^{\mu}\, v^{\nu}, \tag{1}$$

where ϵ and μ are constant parameters that characterize the dielectric and v^{μ} is the four-vector of the medium. Later it was recognized that this interpretation can be used to describe nonlinear structures when ϵ and μ depends on the intensity of the electromagnetic field [3].

In recent years an intense activity concerning properties of Riemannian geometries similar to the one described by Gordon has been done [4]. In particular, those that allow a binomial form for both the metric and its inverse, that is its covariant and the corresponding contravariant expressions are given by

$$\widehat{q}_{\mu\nu} = A\,\gamma_{\mu\nu} + B\,\Phi_{\mu\nu}, \tag{2}$$

and

$$\widehat{q}^{\mu\nu} = \gamma^{\mu\nu} + b\,\Phi^{\mu\nu}, \tag{3}$$

where A, B and b are arbitrary functions and $\gamma_{\mu\nu}$ is the Minkowski metric in an arbitrary coordinate system. Thus, the tensor $\Phi^{\mu\nu}$ must satisfy the condition

$$\Phi_{\mu\nu}\,\Phi^{\nu\lambda} = \alpha\,\delta^{\lambda}_{\mu} + \beta\,\Phi^{\lambda}_{\mu}, \tag{4}$$

where α, β are arbitrary functions and the indexes of $\Phi^{\mu}{}_{\nu}$ are lowered and raised by the background metric $\gamma_{\mu\nu}$. In the present paper we limit our analysis to the simplest dragged form by setting $\Phi^{\mu\nu} = v^{\mu}\,v^{\nu}$. In this case the coefficients of the covariant form of the metric are given by

$$A = 1; \quad B = -\frac{b}{1+b}.$$

The origin of the dragging effect in the case of Gordon's metric is due to the modifications of the path of the electromagnetic waves inside the moving dielectric. Then we face the question: could such particular description of the electromagnetic waves in moving bodies be generalized for other cases, independently of the electromagnetic forces? In other words, could such geometrized paths be used to describe other kinds of force? We shall see that the answer is yes. Indeed, we will show that it is possible to geometrize different kinds of force by the introduction of a dragged metric $\widehat{q}_{\mu\nu}$ such that in this geometry the accelerated body follows the free path of geodesics.

Let us emphasize that we deal here with any kind of force that has a non-gravitational character. It is precisely the consequences of such non-gravitational force that we describe in terms of a modified dragged metric. This means that the observable effects of any force can be interpreted as nothing but a modification of the geometry of space-time. In other words the motion of any accelerated body can be described as a free body following geodesics in a dragged metric. This procedures generalizes d'Alembert principle of classical mechanics [5,6] which states that it is possible to transform a dynamical problem into a static one, where the body is free of the action of any force. Starting from the background metric—where an accelerated body experiences a non-gravitational force—to a dragged metric where the body follows a geodesic and becomes free of any non-gravitational force is the relativistic expression of this principle. In this way we produce a geometric description of all kind of motion whatever the force that originates it.

2 A special case

We claim that accelerated bodies in a flat Minkowski spacetime[1] can be equivalently described as free bodies following geodesics in an associated dragged metric. In order to simplify the calculations we restrict ourselves to the case in which the acceleration vector a_μ is the gradient of a function, that is[2]

$$a_\mu = \partial_\mu \Psi. \tag{5}$$

The force acting on the body under consideration thus comes from a potential V in the Newtonian sense, i.e., $F_\mu = -\partial_\mu V$.

We write the dragged metric in the form

$$\widehat{q}^{\mu\nu} = \eta^{\mu\nu} + b\, v^\mu\, v^\nu,$$

where, in this section, we assumed the Minkowski metric $\eta^{\mu\nu}$ in Lorentzian coordinates. The associated covariant derivative is defined by

$$v^\alpha{}_{;\,\mu} = v^\alpha{}_{,\,\mu} + \widehat{\Gamma}^\alpha_{\mu\nu}\, v^\nu,$$

where the corresponding Christoffel symbol is given by

$$\widehat{\Gamma}^\epsilon_{\mu\nu} = \frac{1}{2}\,(\eta^{\epsilon\alpha} + b\, v^\epsilon\, v^\alpha)\,(\widehat{q}_{\alpha\mu,\nu} + \widehat{q}_{\alpha\nu,\mu} - \widehat{q}_{\mu\nu,\alpha}). \tag{6}$$

1 Let us point out that the analysis we present in this section may be straightforwardly generalized to arbitrary curved Riemannian background.

2 Using the freedom in the definition of the four-vector v^μ, we set $\eta_{\mu\nu} v^\mu v^\nu = 1$. The acceleration is orthogonal to it, that is, $a_\mu\, v^\mu = 0$. We note that we are dealing with a collection of paths Γ that is usually called a congruence of curves. It is understood that each element of this collection concerns particles that have the same characteristics. For instance, if the acceleration is due to electromagnetic field, all particles of Γ must have the same charge-mass ratio, to wit a bunch of electrons.

We use comma (,) to denote simple derivative and hat (ˆ) to denote objects constructed by $\hat{q}^{\mu\nu}$. The description of an accelerated curve in a flat spacetime as a geodesic in a dragged metric is possible if the following condition is satisfied

$$\left(v_{\mu,\nu} - \widehat{\Gamma}^{\epsilon}_{\mu\nu}\, v_{\epsilon}\right)\, \widehat{v}^{\nu} = 0, \tag{7}$$

where we have used the dragged metric to write $\widehat{v}^{\mu} \equiv \widehat{q}^{\mu\nu}\, v_{\nu}$. Or, equivalently,

$$\left(v_{\mu,\nu} - \widehat{\Gamma}^{\epsilon}_{\mu\nu}\, v_{\epsilon}\right)\, v^{\nu} = 0, \tag{8}$$

where $\hat{v}^{\mu} = (1+b)v^{\mu}$. Then, noting that the acceleration in the background is defined by $a_{\mu} = v_{\mu,\nu}\, v^{\nu}$ and using Eq. (5) the condition of geodesic motion in the dragged metric takes the form

$$\partial_{\mu}\Psi = \widehat{\Gamma}^{\epsilon}_{\mu\nu}\, v_{\epsilon}\, v^{\nu}. \tag{9}$$

The right-hand-side of this equation is

$$\widehat{\Gamma}^{\epsilon}_{\mu\nu}\, v_{\epsilon}\, v^{\nu} = \frac{1+b}{2}\, v^{\alpha}\, v^{\nu}\, \widehat{q}_{\alpha\nu,\mu}. \tag{10}$$

Using the expression of $\widehat{q}_{\alpha\nu}$ in this equation and combining the with condition (8), it follows that

$$a_{\mu} + \frac{\partial_{\mu} b}{2(1+b)} = 0,$$

that is, the expression of the coefficient b of the dragged metric is given in terms of the potential of the acceleration

$$1 + b = e^{-2\Psi}. \tag{11}$$

Substituting this equation in Eq. (3), we obtain

$$\widehat{q}^{\mu\nu} = \gamma^{\mu\nu} + (e^{-2\Psi} - 1)\, v^{\mu} v^{\nu}. \tag{12}$$

This simple metric describes the trajectory of v_{μ} as a geodesic motion.

2.1 The curvature of the dragged metric

If the background metric is not flat Minkowski spacetime $\eta_{\mu\nu}$ or we are using an arbitrary coordinate system, the affine connection is given by the sum of the corresponding background one and a tensor, that is

$$\widehat{\Gamma}^{\epsilon}_{\mu\nu} = \Gamma^{\epsilon}_{\mu\nu} + K^{\epsilon}_{\mu\nu}. \tag{13}$$

In the case of the Minkowski background $\eta_{\mu\nu}$ a direct calculation gives for the affine connection the form

$$\widehat{\Gamma}^{\epsilon}_{\mu\nu} \equiv K^{\epsilon}_{\mu\nu} = v^{\epsilon}\,(a_{\mu}\,v_{\nu} + a_{\nu}\,v_{\mu}) - a^{\epsilon}\,v_{\mu}\,v_{\nu}, \tag{14}$$

where we have considered Lorentzian coordinates just to simplify the calculations. Therefore,

$$K^{\epsilon}_{\mu\epsilon} = a_{\mu}.$$

The contracted Ricci curvature has the following expression

$$\widehat{R}_{\mu\nu} = a_{\mu,\nu} - a_{\mu}\,a_{\nu} + (\omega + a^{\alpha}{}_{,\alpha})\,v_{\mu}\,v_{\nu}, \tag{15}$$

Noting that $\widehat{a}^{\mu} = a_{\nu}\,\widehat{q}^{\mu\nu} = a^{\mu}$, we obtain

$$-a^2 \equiv a^{\mu}a_{\mu} = a_{\mu}\,a_{\nu}\,\eta^{\mu\nu} = a_{\mu}\,a_{\nu}\,\widehat{q}^{\mu\nu} = -\widehat{a}^2.$$

The scalar curvature $\widehat{R} = \widehat{R}_{\mu\nu}\,\widehat{q}^{\mu\nu}$ is given by

$$\widehat{R} = (2+b)\,a^{\alpha}{}_{,\alpha}. \tag{16}$$

These expressions can be rewritten in a covariant way remarking that

$$a_{\mu;\nu} \equiv a_{\mu,\nu} - \widehat{\Gamma}^{\epsilon}_{\mu\nu}\,a_{\epsilon},$$

that is, the Ricci curvature is

$$\widehat{R}_{\mu\nu} = a_{\mu\,;\,\nu} - a_{\mu}\,a_{\nu} + (a^2 + a^{\alpha}{}_{;\,\alpha})\,v_{\mu}\,v_{\nu}, \tag{17}$$

and the scalar curvature takes the form

$$\widehat{R} = (2+b)\,(a^{\alpha}{}_{;\alpha} + a^2). \tag{18}$$

Remark the through the whole section we are considering normalized four-vectors only with acceleration in its covariant derivative decomposition. The other kinematical quantities make the calculations more complicated and that is why they will not be considered in this paper (except the vorticity that appears briefly in the last sections).

2.2 Analog gravity

Suppose that an observer following a path with four-velocity v_{μ} and acceleration a_{μ} in the Minkowski spacetime background is not able to identify the origin of the force that acts on him. In other words, the observer is going to believe that only long-range gravitational forces are constraining his motion. Let us assume that he knows that gravity does not accelerate any curve but instead change the metric of the background according to the principles of general relativity. This means that if he is able to represent his motion as a geodesics in a dragged metric $\widehat{q}_{\mu\nu}$ he will consider that the origin of such curved metric is nothing but a consequence of a distribution of energy which he will describe by using the general relativity equations

$$\widehat{R}_{\mu\nu} - \frac{1}{2}\,\widehat{R}\,\widehat{q}_{\mu\nu} = -\widehat{T}_{\mu\nu}. \tag{19}$$

where we set Einstein's constant equal to 1.

He will identify the different terms of the source through his own motion. From his four-velocity, he defines the normalized four-velocity $\hat{u}^{\alpha}$ in the metric $\hat{q}_{\mu\nu}$ by setting

$$\hat{u}^{\alpha} = \sqrt{1+b}\,v^{\alpha}.$$

He then proceed to characterize the origin of the curved metric using the standard decomposition

(a) density of energy

$$\widehat{\rho} = \widehat{T}_{\mu\nu}\,\hat{u}^{\mu}\,\hat{u}^{\nu}; \tag{20}$$

(b) isotropic pressure

$$\widehat{p} = -\frac{1}{3}\,\widehat{T}_{\mu\nu}\,\widehat{h}^{\mu\nu}; \tag{21}$$

(c) heat flux

$$\widehat{q}_{\lambda} = \widehat{T}_{\alpha\beta}\,\hat{u}^{\beta}\,\widehat{h}^{\alpha}{}_{\lambda}; \tag{22}$$

(d) anisotropic pressure

$$\widehat{\pi}_{\mu\nu} = \widehat{T}_{\alpha\beta}\,\widehat{h}^{\alpha}{}_{\mu}\,\widehat{h}^{\beta}{}_{\nu} + \widehat{p}\,\widehat{h}_{\mu\nu}. \tag{23}$$

In these expressions we used

$$\widehat{h}_{\mu\nu} = \widehat{q}_{\mu\nu} - \widehat{u}_{\mu}\,\widehat{u}_{\nu}.$$

Note that $\widehat{h}_{\mu\nu} = h_{\mu\nu}$. We thus write

$$\widehat{T}_{\mu\nu} = \widehat{\rho}\,\hat{u}_{\mu}\,\hat{u}_{\nu} - \widehat{p}\,h_{\mu\nu} + \widehat{q}_{\mu}\,\hat{u}_{\nu} + \widehat{q}_{\nu}\,\hat{u}_{\mu} + \widehat{\pi}_{\mu\nu}. \tag{24}$$

From this decomposition, using Eq. (19) and the Ricci curvature (17) we identify the energy-momentum components as

$$\begin{aligned}
\widehat{\rho} &= \frac{b}{2}\,Q,\\
\widehat{p} &= \left(\frac{2}{3}+\frac{b}{2}\right)\,Q,\\
\widehat{q}_{\mu} &= 0,\\
\widehat{\pi}_{\mu\nu} &= -\,a_{\mu\,;\,\nu} + a_{\mu}\,a_{\nu} - \frac{Q}{3}\,\widehat{q}_{\mu\nu} + \frac{Q}{3}\,\hat{u}_{\mu}\,\hat{u}_{\nu},
\end{aligned} \tag{25}$$

where

$$Q = -(a^2 + a^{\alpha}{}_{;\,\alpha}).$$

Summarizing, we can say that this observer will state that there is a gravitational field represented by the metric $\widehat{q}_{\mu\nu}$ produced by the energy-momentum distribution given by equation (25). We note that this reduction of the dragged metric to the framework of general relativity is not mandatory. Indeed, we deal here precisely with some accelerated paths that are not reduced to the gravitational force in the standard theory. This will become more clear when we present examples of accelerated curves in specific solutions of general relativity in the next sections.

Now let us present some clarifying examples. The first one considers the motion of rotating bodies in Minkowski space-time. The other ones take into account the general relativity effects. We shall see that it is possible to produce what could be called *double gravity*, if the origin of the curvature of the dragged metric is identified to an effective energy-momentum tensor satisfying the general relativity equations. However, there is no reason for this restriction. We will come back to this question elsewhere.

3 Accelerated particles in Minkowski spacetime

Let us consider a simple example concerning the acceleration of a body in flat Minkowski spacetime written in the cylindrical coordinate system (t, r, ϕ, z) to express the following line element

$$ds^2 = a^2[dt^2 - dr^2 - dz^2 + g(r)d\phi^2 + 2h(r)d\phi dt], \tag{26}$$

where a is a constant. We choose the following local tetrad frame given implicitly by the 1-forms

$$\begin{aligned}
\theta^0 &= a(dt + hd\phi),\\
\theta^1 &= adr,\\
\theta^2 &= a\Delta d\phi,\\
\theta^3 &= adz,
\end{aligned} \tag{27}$$

where we define $\Delta = \sqrt{h^2 - g}$. The only non-identically zero components of the Riemann tensor $R^{A}{}_{BCD}$ in the tetrad frame are

$$
\begin{aligned}
R^0{}_{101} &= \frac{1}{4a^2}\left(\frac{h'}{\Delta}\right)^2, \\
R^0{}_{112} &= -\frac{1}{2a^2}\left(\frac{h''}{\Delta} - \frac{h'\Delta'}{\Delta^2}\right), \\
R^0{}_{202} &= \frac{1}{4a^2}\left(\frac{h'}{\Delta}\right)^2, \\
R^1{}_{212} &= \frac{1}{a^2}\left[\frac{\Delta''}{\Delta} - \frac{3}{4}\left(\frac{h'}{\Delta}\right)^2\right],
\end{aligned}
\tag{28}
$$

where prime $'$ means derivative with respect to the radial coordinate r. The equations of general relativity for this geometry have two simple solutions that we shall analyze below.

In the case of $R^A{}_{BCD}$=0, we get

$$h' = 0; \quad \Delta'' = 0.$$

Solving these equations, we find

$$h \equiv const; \quad \Delta \equiv \omega r,$$

where ω is a constant. Therefore, Eq. (26) takes the form

$$ds^2 = a^2[dt^2 - dr^2 - dz^2 + (h^2 - \omega^2 r^2)d\phi^2 + 2h\, d\phi dt]. \tag{29}$$

If we consider the observer field

$$v^\mu = \frac{1}{a\sqrt{h^2 - \omega^2 r^2}}\delta^\mu_2.$$

This path corresponds to an acceleration vector given by

$$a_\mu = \left(0, \frac{\omega^2 r}{(h^2 - \omega^2 r^2)}, 0, 0\right).$$

This means that $a_\mu = \partial_\mu \Psi$, where

$$2\Psi = -\ln(h^2 - \omega^2 r^2).$$

We are in a situation similar to the previous section since the acceleration is a gradient. The parameter b of the dragged metric, given by the expression (11), can be written down as

$$1 + b = h^2 - \omega^2 r^2,$$

and for the dragged metric, we get

$$\frac{ds^2}{a^2} = \frac{\omega^4 r^4 - \omega^2 r^2 h^2 + 1}{(h^2 - \omega^2 r^2)^2} dt^2 + d\phi^2 + \frac{2h}{h^2 - \omega^2 r^2} d\phi\, dt - dr^2 - dz^2. \quad (30)$$

Note that we are dealing with the case $h^2 - \omega^2 r^2 > 0$. This allows the presence of accelerated closed time-like curves (CTC) in the original background that will be mapped into closed time-like geodesics (CTG) in the dragged geometry. Note that there exists a real singularity in $r = h/\omega$ and that $\widehat{q}_{00}$ changes its sign at

$$\omega^2\, r_\pm^2 = \frac{h^2 \pm \sqrt{h^4 - 4}}{2}.$$

4 Accelerated particles in curved spacetimes

Let us present now how our dragged metric approach works in some solutions of the general relativity equations. We choose three well-known geometries: Schwarzschild, Gödel universe and the Kerr solution in which accelerated particles have very peculiar properties. In these Riemannian manifolds we analyze some examples of accelerated paths that are interpreted as geodesics in the associated dragged metrics.

4.1 Schwarzschild geometry

We start with the Schwarzschild metric in the (t, r, θ, φ) coordinate system

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \frac{1}{\left(1 - \frac{r_H}{r}\right)}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (31)$$

and choose the trajectory described by the four-velocity

$$v_\mu = \sqrt{1 - \frac{r_H}{r}}\, \delta^0_\mu.$$

The corresponding acceleration is

$$a_\mu = \left(0, -\frac{r_H}{2(r^2 - r\, r_H)}, 0, 0\right).$$

In this case the acceleration is gradient of the function Ψ given by

$$\Psi = -\frac{1}{2}\, \ln\left(1 - \frac{r_H}{r}\right).$$

The dragged metric coefficient b is given by

$$b = -\frac{r_H}{r},$$

and, therefore, the dragged metric takes the form

$$\widehat{ds}^2 = dt^2 - \frac{1}{\left(1 - \frac{r_H}{r}\right)} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{32}$$

The only non-vanishing Ricci tensor components of the metric $\widehat{q}^{\mu\nu}$ are

$$R^1_1 = \frac{r_H}{r^3};$$
$$R^2_2 = R^3_3 = -\frac{1}{2} R^1_1.$$

According to Eqs. (20)–(23), the energy-momentum tensor corresponding to this dragged metric has only an anisotropic pressure term (which is proportional to the components $C_{0\mu 0\nu}$ of the conformal tensor of this metric). Such fluid has no physical meaning. All Debever invariants are finite in each points except at the origin $r = 0$.

In the following examples we will not present this previous analysis in terms of the energy-momentum tensor because the dragged metric is more complicated resulting in $\widehat{T}_{\mu\nu}$ without simple physical interpretation at first sight. Notwithstanding, it does not interferes the aims of this paper.

4.2 Gödel's geometry

Let us now turn our analysis to the Gödel geometry. In the cylindrical coordinate system this metric is given by Eq. (26), where a is a constant related to the vorticity $a = 2/\omega^2$ and

$$h(r) = \sqrt{2}\sinh^2 r;$$
$$g(r) = \sinh^2 r(\sinh^2 r - 1).$$

For the sake of completeness we note the nontrivial contravariant terms of this metric are

$$g^{00} = \frac{1 - \sinh^2 r}{a^2\, \cosh^2 r},$$
$$g^{02} = \frac{\sqrt{2}}{a^2\, \cosh^2 r}, \tag{33}$$
$$g^{22} = \frac{-1}{a^2\, \sinh^2 r\, \cosh^2 r}.$$

In Ref. [7] it was pointed out the acausal properties of a particle moving into a circular orbit around the z-axis with four-velocity

$$v^\mu = \left(0, 0, \frac{1}{a\, \sinh r\, \sqrt{\sinh^2 r - 1}}, 0\right).$$

This path corresponds to an acceleration given by

$$a^{\mu} = \left(0, \frac{\cosh r\,[2\,\sinh^2 r - 1]}{a^2\,\sinh r\,[\sinh^2 r - 1]}, 0, 0\right).$$

This means that $a_{\mu} = \partial_{\mu}\Psi$, where

$$\Psi = -\ln(\sinh r\,\sqrt{\sinh^2 r - 1}).$$

We are again in the situation where the acceleration is a gradient. Therefore, the parameter b of the dragged metric is given by Eq. (11) and in Gödel's background it takes the form

$$1 + b = \sinh^2 r(\sinh^2 r - 1).$$

The dragged metric has the following expression

$$\frac{\widehat{ds}^2}{a^2} = \frac{3 - \sinh^4 r}{(\sinh^2 r - 1)^2}dt^2 + d\phi^2 + 2\frac{\sqrt{2}}{\sinh^2 r - 1}d\phi\,dt - dr^2 - dz^2. \tag{34}$$

From the analysis of geodesics in Gödel's geometry the domain $r < r_c$ where $\sinh^2 r_c = 1$ separates the causal from the non-causal regions of the spacetime. This is related to the fact that a geodesic that reaches the value $r = 0$ will be confined within the domain Ω_i defined by the region $0 < r < r_c$ (see Ref. [8] for details upon geodesic motion in Gödel's metric). However, the gravitational field is finite in the region $r = r_c$. Nothing similar happens in the dragged metric, since at $\sinh^2 r = 1$ there exists a real singularity in the dragged metric. Only the exterior domain is allowed. This means that for this kind of accelerated path in Gödel geometry the allowed domain for the dragged metric is precisely the whole non-causal region.

4.3 Kerr metric

Let us turn now to the dragged metric approach in the case the background is the Kerr metric. In the Boyer-Lindquist coordinate system this metric is given by

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Sigma}dr^2 - \rho^2 d\theta^2 + \frac{4Mra\sin^2\theta}{\rho^2}dtd\phi$$
$$- \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right]d\phi^2, \tag{35}$$

where $\Sigma = r^2 + a^2 - 2Mr$ and $\rho^2 = r^2 + a^2\cos^2\theta$. On the equatorial plane ($\theta = \pi/2$) consider the following vector field

$$v^{\mu} = \left(0, 0, 0, \frac{r}{\sqrt{-(r^2+a^2)^2 + a^2\Sigma}}\right).$$

This path corresponds to an acceleration given by

$$a_{\mu} = \left(0, -\frac{r^3 - Ma^2}{r^4 + r^2a^2 + 2Mra^2}, 0, 0\right).$$

This means that $a_{\mu} = \partial_{\mu}\Psi$, where

$$2\Psi = -\ln\left[-\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\right].$$

Over again, we choose an accelerated path that can be represented by a gradient. The parameter b is given by

$$1 + b = -\left(r^2 + a^2 + \frac{2Ma^2}{r}\right),$$

and the dragged metric, on the equatorial plane, takes the form

$$\frac{\widehat{ds}^2}{a^2} = \frac{1}{(1+b)^2}\left(1 - \frac{2M}{r} - b\frac{4M^2a^2}{r^2}\right)dt^2 + d\phi^2 + \frac{4Ma}{(1+b)r}dtd\phi - \frac{r^2}{\Delta}dr^2. \quad (36)$$

These last cases (Gödel and Kerr metrics) show a very curious and intriguing property: the accelerated CTC's at their respective metrics) are transformed into geodesic curves, that is CTG's. Moreover, the dragged metrics display a real singularity excluding the causal domain in both cases.

5 General case

In the precedent sections we limited our analysis to the case in which the acceleration is given by a unique function. Let us now pass to more general situation. In order to geometrize any kind of force we must deal with a larger class of geometries. The most general form of dragged metric that allows the description of accelerated bodies as true geodesics in a modified geometry has the form

$$\widehat{q}^{\mu\nu} = g^{\mu\nu} + b\, v^{\mu}\, v^{\nu} + m\, a^{\mu}\, a^{\nu} + n a^{(\mu}\, v^{\nu)}, \quad (37)$$

where we denoted $a^{(\mu}\, v^{\nu)} \equiv a^{\mu}\, v^{\nu} + a^{\nu}\, v^{\mu}$. The three arbitrary parameters b, m, n are related to the three degrees of freedom of the acceleration vector. The corresponding covariant form of the metric is given by

$$\widehat{q}_{\mu\nu} = g_{\mu\nu} + B\, v_{\mu}\, v_{\nu} + M\, a_{\mu}\, a_{\nu} + N a_{(\mu}\, v_{\nu)}, \quad (38)$$

in which B, M, N are given in terms of b, m, n by

$$B = -\frac{b\,(1 - m\,a^2) + n^2\,a^2}{(1+b)\,(1 - m\,a^2) + n^2\,a^2},$$
$$M = \frac{1}{1 - m\,a^2}\left(-m + \frac{n^2}{(1+b)\,(1 - m\,a^2) + n^2\,a^2}\right),$$

and

$$N = -\frac{n}{(1+b)\,(1 - m\,a^2) + n^2\,a^2}.$$

In this case the equation that generalizes the geodesic condition (8) takes the form

$$a_\mu = \frac{1}{2}\left[(1+b)\,v^\lambda\,v^\nu + n\,a^\lambda\,a^\nu\right]\,(\widehat{q}_{\lambda\mu\,,\,\nu} + \widehat{q}_{\lambda\nu\,,\,\mu} - \widehat{q}_{\mu\nu\,,\,\lambda}). \tag{39}$$

This equation can be cast in the following formal expression

$$a_\mu = -\frac{b_{,\mu}}{2(1+b)} - n\omega_{\mu\nu}a^\nu, \tag{40}$$

where $\omega_{\mu\nu} \equiv v_{[\mu,\nu]} - a_{[\mu}v_{\nu]}$ is the vorticity tensor. Solving this equation for these functions provides the most general expression for any acceleration.

With these results, we have transformed the path of any particle submitted to any kind of force as a geodetic motion in the dragged metric. This result is the extension of the d'Alembert principle, corresponding to all types of motion, i.e., the acceleration is geometrized through the dragged metric approach.

6 Distinct accelerated particles in the same dragged metric

In this section we shall present a systematic way to encounter other accelerated vector fields that satisfy the same requirements to follow a geodesic motion in the same dragged metric that a previously given vector field. For it, consider an accelerated congruence v^μ in Minkowski spacetime following a geodetic motion in the dragged metric

$$\hat{q}^{\mu\nu} = \eta^{\mu\nu} + b\,v^\mu v^\nu + m\,a^\mu a^\nu + n\,a^{(\mu}v^{\nu)}. \tag{41}$$

We have shown that

$$a_\mu = \partial_\mu\Psi - n\omega_{\mu\nu}a^\nu,$$

where $1 + b = e^{-2\Psi}$. Note that the intrinsic properties (mass, charge etc.) of the particle represented by v^μ are contained in Ψ and n.

Now, consider another four-vector $\tilde{v}^\mu$ such that it follows a geodetic motion in a dragged metric

$$\hat{q}^{\mu\nu}_{(\tilde{v})} = \eta^{\mu\nu} + \tilde{b}\,\tilde{v}^{\mu}\tilde{v}^{\nu}. \tag{42}$$

For convenience, suppose that $\tilde{v}^{\mu}\tilde{v}_{\mu} = 1$ and $\tilde{v}_{\mu,\nu} = \tilde{a}_{\mu}\tilde{v}_{\nu}$. This last assumption implies that $\tilde{a}_{\mu} = \partial_{\mu}\tilde{\Psi}$, where $\tilde{\Psi}$ contains all information about kinematical features of $\tilde{v}^{\mu}$. We then ask if it is possible that these congruences v^{μ} and $\tilde{v}^{\mu}$ could agree with the same metric structure. In principle, we suppose that $\tilde{v}^{\mu}$ lies on the plane generated by v^{μ} and a^{μ}, i.e.,

$$\tilde{v}^{\mu} = p\,v^{\mu} + q\,a^{\mu}, \tag{43}$$

where p and q are arbitrary functions. As we set $\tilde{v}^{\mu}\tilde{v}_{\mu} = 1$, this restricts the coefficients p and q to $p^2 - q^2a^2 = 1$. Therefore, from the equation of motion for $\tilde{v}^{\mu}$, i.e., $\tilde{v}_{\mu,\nu}\tilde{v}^{\nu} = \tilde{a}_{\mu}$, we obtain

$$\tilde{a}_{\mu} = (p\,\dot{p} + q\,p' + p\,q\,a^2)v_{\mu} + \left[2 - p^2 + \frac{p}{q\,a^2}(p\,\dot{p} + q\,p')\right]a_{\mu}. \tag{44}$$

where $\dot{p} \equiv p_{,\mu}v^{\mu}$ and $p' \equiv p_{,\mu}a^{\mu}$. Now we look for the case $\tilde{a}_{\mu} \propto a_{\mu}$ which implies that $p\dot{p} + qp' + pqa^2 = 0$. This assumption is made in order to answer if it is possible that different particles subjected to the same force on the background can follow a geodetic motion in the same dragged metric. Under these considerations, Eq. (44) yields

$$\tilde{a}_{\mu} = 2(1 - p^2)a_{\mu}. \tag{45}$$

Eq. (45) implies that

$$\partial_{\mu}\tilde{\Psi} = 2(1 - p^2)(\partial_{\mu}\Psi - n\omega_{\mu\nu}a^{\nu}). \tag{46}$$

Note that the potential $\tilde{\Psi}$ must satisfy this equation in order that the acceleration of v^{μ} be proportional to the acceleration of $\tilde{v}^{\mu}$. If they follow geodesics in the same geometry, then an extra proposition must be satisfied

$$\hat{q}^{\mu\nu}_{(\tilde{v})} = \hat{q}^{\mu\nu} \Longrightarrow \tilde{b}\,\tilde{v}^{\mu}\tilde{v}^{\nu} = b\,v^{\mu}v^{\nu} + m\,a^{\mu}a^{\nu} + n\,a^{(\mu}v^{\nu)}. \tag{47}$$

Substituting (43) into (47), we obtain

$$\begin{aligned} p &= \sqrt{\frac{e^{-2\Psi} - 1}{e^{-2\tilde{\Psi}} - 1}}, \\ q &= n\left[(e^{-2\Psi} - 1)(e^{-2\tilde{\Psi}} - 1)\right]^{-1/2}, \\ m &= \frac{n^2}{e^{-2\Psi} - 1}. \end{aligned} \tag{48}$$

Remark that we still have an arbitrariness in the choice of n. Once it is fixed, then we can uniquely determine the dragged metric in which v^{μ} and $\tilde{v}^{\mu}$ follow geodesics. This could be the case of particles in a electromagnetic field with the same charge-mass ratio. This case has several developments and we will analyze with more details in a future work.

7 Conclusion

We summarize the novelty of our analysis in the following steps:

- Let v_{μ} represent the four-vector that describes the kinematics of a body in an arbitrary spacetime endowed with a geometry $g_{\mu\nu}$;
- If the body experiences a non-gravitational force it acquires an acceleration a_{μ};
- It is always possible to define an associated dragged metric $\widehat{q}_{\mu\nu}$ given by (38) such that in this metric the acceleration is removed. That is, the particle trajectory is represented as a geodesic in $\widehat{q}_{\mu\nu}$.

We have shown by a constructive operation that the following result is true:

Lemma *For any accelerated path* Γ *described by four-vector velocity* v_{μ} *and acceleration* a_{μ} *in a given Riemannian geometry* $g_{\mu\nu}$ *we can always construct another geometry* $\widehat{q}$ *endowed with a dragged metric* $\widehat{q}_{\mu\nu}$ *which depends only on* $g_{\mu\nu}$, v_{μ} *and* a_{μ} *such that the path* Γ *is a geodesic in* $\widehat{q}$.

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