Braz J Phys (2014) 44:832–894
DOI 10.1007/s13538-014-0239-1

SOCIEDADE BRASILEIRA DE FÍSICA

PARTICLES AND FIELDS

The Quasi-Maxwellian Equations of General Relativity: Applications to Perturbation Theory

M. Novello · E. Bittencourt · J. M. Salim

Received: 24 March 2014 / Published online: 8 August 2014

Abstract A comprehensive review of the equations of general relativity in the quasi-Maxwellian (QM) formalism introduced by Jordan, Ehlers and Kundt is presented. Our main interest concerns its applications to the analysis of the perturbation of standard cosmology in the Friedman-Lemaître-Robertson-Walker framework. The major achievement of the QM scheme is its use of completely gauge-independent quantities. We shall see that in the QM-scheme, we deal directly with observable quantities. This reveals its advantage over the old method introduced by Lifshitz that deals with perturbation in the standard framework. For completeness, we compare the QM-scheme to the gauge-independent method of Bardeen, a procedure consisting of particular choices of the perturbed variables as a combination of gauge-dependent quantities.

Keywords Quasi-Maxwellian equations · Cosmology · Perturbation theory

M. Novello is Cesare Lattes ICRANet Professor

M. Novello · E. Bittencourt (✉) · J. M. Salim
Instituto de Cosmologia Relatividade Astrofisica ICRA—CBPF,
Rua Dr. Xavier Sigaud, 150, CEP 22290-180,
Rio de Janeiro, Brazil
e-mail: eduardo.bittencourt@icranet.org

M. Novello
e-mail: novello@cbpf.br

J. M. Salim
e-mail: jsalim@cbpf.br

E. Bittencourt
Sapienza Università di Roma—Dipartimento di Fisica,
P.le Aldo Moro 5, 00185 Rome, Italy

E. Bittencourt
ICRANet, Piazza della Repubblica 10, 65122 Pescara, Italy

1 Introduction

There are two formal ways to deal with the dynamics of General Relativity, which we call the Einstein frame and the Jordan-Ehlers-Kundt frame (JEK frame, for short):

- The Einstein frame (1915) corresponds to a second-order differential equations relating the curvature tensor to the energy-momentum tensor
- The JEK frame [79, 80] relates the derivatives of the conformal Weyl tensor to the derivatives of the energy-momentum tensor using Bianchi's identities.

Although the JEK formulation was proven equivalent to general relativity (GR) by Lichnerowicz in the early 1960s, its role in the development of applications have been less active than one could expect. This may have happened because almost all introductory books on GR fail to present the JEK frame as an alternative formulation of gravitation. Indeed, only a few advanced books—cf. Zakharov [164], Choquet-Bruhat [35] and Hawking-Ellis [64]—show an overview on this. In particular, as a direct consequence, the great majority of perturbation-theory analyses completely ignores the possibility of using the JEK frame to develop a consistent, worldwide method.

The main goal of the present review is to make the JEK frame a little more popular in the realm of Friedman-universe perturbation theory. Indeed, the Lifshitz-Bardeen method and the JEK frame, under the same initial conditions, give the same results for perturbations in the linear regime, as we shall see in Section 3.4.3. The main interest in JEK rests on its unambiguous way to deal with perturbation within the standard cosmological FLRW scenario.

Alternatively, the JEK-frame is called the quasi-Maxwellian version of general relativity. The reason for this name is manifest, given its striking similitude to Maxwell's equations of electrodynamics as can be seen in Marteens and Bassett [92]. The Appendix exploits this similarity to exhibit an example of modification of general relativity by extending further the analogy to the case of the electrodynamics inside a dielectric medium.

This paper is summarized as follows: we concentrate our attention on the (cosmological) perturbation scheme presented in Section 3, although we will describe some examples of well-known solutions—Schwarzschild, Kasner and Friedman (singular and nonsingular)—according to the JEK frame (cf. Section 2) to show how this method could be used to obtain new solutions of general relativity (details of this discussion in Section 4).

1.1 Definitions, Notations and a Brief Mathematical Compendium

We start out with a list of the definitions in this review:

- The structure of the space-time is represented by a Riemannian geometry $g_{\mu\nu}(x^\alpha)$, with Lorentzian signature $(+,-,-,-)$;
- The Levi-Civita tensor $\eta_{\mu\nu\alpha\beta} = \sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta}$, where g is the determinant of $g_{\mu\nu}$ and $\epsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric pseudo-tensor, $\epsilon_{0123} = 1$;
- The Christoffel symbols are defined by $\Gamma^\alpha_{\beta\mu} = \frac{1}{2} g^{\alpha\lambda}(g_{\beta\lambda,\mu} + g_{\mu\lambda,\beta} - g_{\beta\mu,\lambda})$;
- The geodesic equation is
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0;$$
- The Riemann tensor is defined by
$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\mu,\nu} - \Gamma^\alpha_{\beta\nu,\mu} + \Gamma^\alpha_{\nu\tau}\Gamma^\tau_{\beta\mu} - \Gamma^\alpha_{\mu\tau}\Gamma^\tau_{\beta\nu};$$
The traces of Riemann tensor define the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and the curvature scalar $R = R^\alpha{}_\alpha$;
- The decomposition of the energy-momentum tensor into irreducible parts, with respect to a normalized observer field V^α, is given by
$$T_{\mu\nu} = \rho V_\mu V_\nu - p h_{\mu\nu} + V_{(\mu} q_{\nu)} + \pi_{\mu\nu},$$
where ρ is the energy density, p is the isotropic pressure, q_μ is the heat flux, and $\pi_{\mu\nu}$ is the anisotropic pressure. We use parentheses "( )" for symmetrization and brackets "[ ]" for skew-symmetrization.
- The Weyl tensor is defined by
$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6} R g_{\alpha\beta\mu\nu},$$
where
$$2M_{\alpha\beta\mu\nu} = R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}$$
and
$$g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}.$$
The dual is denoted by
$$W^*_{\alpha\beta\mu\nu} = \frac{1}{2}\eta_{\alpha\beta}{}^{\rho\sigma} W_{\rho\sigma\mu\nu}.$$
Note that $W^*_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu}{}^*$.
- The electric and magnetic parts of the Weyl tensor are, respectively,
$$E_{\alpha\beta} \equiv -W_{\alpha\mu\beta\nu} V^\mu V^\nu$$
and
$$H_{\alpha\beta} \equiv -{}^*W_{\alpha\mu\beta\nu} V^\mu V^\nu.$$
- The tensor defined by $h_{\mu\nu} \equiv g_{\mu\nu} - V_\mu V_\nu$ projects tensorial quantities in the rest space Σ of the observers. Note that $h_{\mu\nu} V^\nu = 0$ and $h_{\mu\nu} h^\nu{}_\lambda = h_{\mu\lambda}$.
- Einstein's equations (EE) are given by
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -k T_{\mu\nu},$$
where Λ is the cosmological constant and $k \equiv 8\pi G_N/c^4$, which we shall set equal to 1, unless stated otherwise.
- The covariant derivative of V_μ can be decomposed into its irreducible parts, that is,
$$V_{\mu;\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} + a_\mu V_\nu,$$
where $\theta = V^\alpha{}_{;\alpha}$ is the expansion coefficient,
$$\sigma_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu V_{(\alpha;\beta)} - \frac{\theta}{3} h_{\mu\nu}$$
is the shear tensor and
$$\omega_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu V_{[\alpha;\beta]}$$
is the vorticity and a^μ is the acceleration.

One can use the above-defined quantities to obtain the evolution equations of the kinematical quantities:

- Raychaudhuri equation
$$\dot\theta + \frac{\theta^2}{3} + 2\sigma^2 + 2\omega^2 - a^\mu{}_{;\mu} = -\frac{1}{2}(\rho + 3p), \qquad (1)$$
where $2\sigma^2 \equiv \sigma^{\mu\nu}\sigma_{\mu\nu}$ and $2\omega^2 \equiv \omega^{\mu\nu}\omega_{\mu\nu}$ and $\dot X \equiv X_{,\alpha} V^\alpha$ (this last definition will be used throughout the text).
- The evolution of the shear tensor is
$$h_\alpha{}^\mu h_\beta{}^\nu \dot\sigma_{\mu\nu} + \frac{1}{3} h_{\alpha\beta}\left(a^\lambda{}_{;\lambda} - 2\sigma^2 - 2\omega^2\right) + a_\alpha a_\beta - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu (a_{\mu;\nu} + a_{\nu;\mu}) + \frac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^\mu{}_\beta + \omega_{\alpha\mu}\omega^\mu{}_\beta = -E_{\alpha\beta} - \frac{1}{2}\pi_{\alpha\beta}. \qquad (2)$$

- The evolution equation for the vorticity tensor is given by the expression

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\omega}_{\mu\nu} - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu (a_{\mu;\nu} - a_{\nu;\mu}) + \frac{2}{3}\theta\omega_{\alpha\beta} - \sigma_{\beta\mu}\omega^\mu{}_\alpha + \sigma_{\alpha\mu}\omega^\mu{}_\beta = 0; \tag{3}$$

- These kinematical quantities must satisfy three constraint equations:

$$\frac{2}{3}\theta_{,\mu} h^\mu{}_\lambda - \left(\sigma^\alpha{}_\gamma + \omega^\alpha{}_\gamma\right)_{;\alpha} h^\gamma{}_\lambda - a^\nu(\sigma_{\lambda\nu} + \omega_{\lambda\nu}) = -q_\lambda; \tag{4}$$

$$\omega^\alpha{}_{;\alpha} + 2\omega^\alpha a_\alpha = 0, \tag{5}$$

where $\omega^\alpha = -\frac{1}{2}\eta^{\alpha\beta\gamma\delta}\omega_{\beta\gamma} V_\delta$ and

$$-\frac{1}{2} h_{(\tau}{}^{\epsilon} h_{\lambda)}{}^{\alpha} \eta_\epsilon{}^{\beta\gamma\nu} V_\nu (\sigma_{\alpha\beta} + \omega_{\alpha\beta})_{;\gamma} + a_{(\tau}\omega_{\lambda)} = H_{\tau\lambda}; \tag{6}$$

- The conservation law of the energy-momentum tensor expressed in terms of its components is the conservation equation

$$\dot{\rho} + (\rho + p)\theta + \dot{q}^\mu V_\mu + q^\alpha{}_{;\alpha} - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{7}$$

and the generalized Euler equation

$$(\rho + p) a_\alpha - p_{,\mu} h^\mu{}_\alpha + \dot{q}_\mu h^\mu{}_\alpha + \theta q_\alpha + q^\nu \theta_{\alpha\nu} + q^\nu \omega_{\alpha\nu} + \pi_\alpha{}^\nu{}_{;\nu} + \pi^{\mu\nu}\sigma_{\mu\nu} V_\alpha = 0. \tag{8}$$

These formulas summarize the kinematical part of the QM-approach. Now, we shall focus on the dynamical equations in terms of the Weyl tensor.

1.2 Quasi-Maxwellian Equations

The quasi-Maxwellian equations [65, 79, 80, 122, 141] are obtained from Bianchi's identities written in terms of the Weyl tensor, i.e., from the expression

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha} R^{,\beta]}. \tag{9}$$

Substitution of the Einstein equations on the right-hand side then yields the equality

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha} T^{,\beta]}. \tag{10}$$

From a practical viewpoint, we find it convenient to project these equations with reference to a vector field V^α and its orthogonal hyper-surface of spatial metric $h_{\mu\nu}$. There are four possibilities to do the decomposition and, therefore, four linearly independent equations:

- The projection $V_\beta V_\mu h_\alpha{}^\sigma$ gives

$$h^{\epsilon\alpha} h^{\lambda\gamma} E_{\alpha\lambda;\gamma} + \eta^\epsilon{}_{\beta\mu\nu} V^\beta H^{\nu\lambda}\sigma^\mu{}_\lambda + 3H^{\epsilon\nu}\omega_\nu = \frac{1}{3} h^{\epsilon\alpha}\rho_{,\alpha} + \frac{\theta}{3} q^\epsilon + -\frac{1}{2}(\sigma^\epsilon{}_\nu - 3\omega^\epsilon{}_\nu) q^\nu + \frac{1}{2}\pi^{\epsilon\mu} a_\mu + \frac{1}{2} h^{\epsilon\alpha}\pi_\alpha{}^\nu{}_{;\nu}. \tag{11}$$

- The projection $\eta^{\sigma\lambda}{}_{\alpha\beta} V_\mu V_\lambda$ yields

$$h^{\epsilon\alpha} h^{\lambda\gamma} H_{\alpha\lambda;\gamma} - \eta^\epsilon{}_{\beta\mu\nu} V^\beta E^{\nu\lambda}\sigma^\mu{}_\lambda - 3E^{\epsilon\nu}\omega_\nu = (\rho + p)\omega^\epsilon - \frac{1}{2}\eta^{\epsilon\alpha\beta\lambda} V_\lambda q_{\alpha;\beta} + \frac{1}{2}\eta^{\epsilon\alpha\beta\lambda}(\sigma_{\mu\beta} + \omega_{\mu\beta})\pi^\mu{}_\alpha V_\lambda; \tag{12}$$

- The projection $h_\mu{}^{(\sigma}\eta^{\tau)\lambda}{}_{\alpha\beta} V_\lambda$ gives

$$\begin{aligned} & h_\mu{}^\epsilon h_\nu{}^\lambda \dot{H}^{\mu\nu} + \theta H^{\epsilon\lambda} - \frac{1}{2} H_\nu{}^{(\epsilon} h^{\lambda)}{}_\mu V^{\mu;\nu} \\ & + \eta^{\lambda\nu\mu\gamma}\eta^{\epsilon\beta\tau\alpha} V_\mu V_\tau H_{\alpha\gamma}\theta_{\nu\beta} + \\ & - a_\alpha E_\beta{}^{(\lambda}\eta^{\epsilon)\gamma\alpha\beta} V_\gamma + \frac{1}{2} E_\beta{}^\mu{}_{;\alpha} h_\mu^{(\epsilon}\eta^{\lambda)\gamma\alpha\beta} V_\gamma \\ & = -\frac{3}{4} q^{(\epsilon}\omega^{\lambda)} + \frac{1}{2} h^{\epsilon\lambda} q^\mu \omega_\mu + \\ & + \frac{1}{4}\sigma_\beta{}^{(\epsilon}\eta^{\lambda)\alpha\beta\mu} V_\mu q_\alpha + \frac{1}{4} h^{\nu(\epsilon}\eta^{\lambda)\alpha\beta\mu} V_\mu \pi_{\nu\alpha;\beta}; \end{aligned} \tag{13}$$

- The projection $V_\beta h_{\mu(\tau} h_{\sigma)\alpha}$ yields

$$\begin{aligned} & h_\mu{}^\epsilon h_\nu{}^\lambda \dot{E}^{\mu\nu} + \theta E^{\epsilon\lambda} - \frac{1}{2} E_\nu{}^{(\epsilon} h^{\lambda)}{}_\mu V^{\mu;\nu} \\ & + \eta^{\lambda\nu\mu\gamma}\eta^{\epsilon\beta\tau\alpha} V_\mu V_\tau E_{\alpha\gamma}\theta_{\nu\beta} + \\ & + a_\alpha H_\beta{}^{(\lambda}\eta^{\epsilon)\gamma\alpha\beta} V_\gamma - \frac{1}{2} H_\beta{}^\mu{}_{;\alpha} h_\mu^{(\epsilon}\eta^{\lambda)\gamma\alpha\beta} V_\gamma \\ & = \frac{1}{6} h^{\epsilon\lambda}(q^\mu{}_{;\mu} - q^\mu a_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}) + \\ & - \frac{1}{2}(\rho + p)\sigma^{\epsilon\lambda} + \frac{1}{2} q^{(\epsilon} a^{\lambda)} - \frac{1}{4} h^{\mu(\epsilon} h^{\lambda)\alpha} q_{\mu;\alpha} \\ & + \frac{1}{2} h_\alpha{}^\epsilon h_\mu{}^\lambda \dot{\pi}^{\alpha\mu} + \frac{1}{4}\pi_\beta{}^{(\epsilon}\sigma^{\lambda)\beta} + \\ & - \frac{1}{4}\pi_\beta{}^{(\epsilon}\omega^{\lambda)\beta} + \frac{1}{6}\theta\pi^{\epsilon\lambda}. \end{aligned} \tag{14}$$

Equations (11)–(14) are the QM equations. We will now show that the QM-formalism is consistent and equivalent to the dynamics of general relativity.

1.3 Equivalence between QM Equations and GR

The QM formalism is supported by the theorems mentioned in this section. While the quoted references are mostly interested in its formal aspects, here we shall focus on the physical results.

Following the steps taken by Lichnerowicz [88] to prove the equivalence between QM equations and GR, we first consider a manifold $\mathcal{M}$ with $n+1$ dimensions endowed with a hyperbolic metric $g_{\mu\nu}$ satisfying Einstein's equations and assume that a hyper-surface Σ has the local equation $f(x^\alpha)=0$. We assume that the discontinuity of the derivatives of $g_{\mu\nu}$ when it crosses Σ is given by Hadamard's conditions, i.e.,

$$[g_{\mu\nu,\alpha,\beta}]_\Sigma = a_{\mu\nu} f_{,\alpha} f_{,\beta}, \tag{15}$$

where the amplitude of the discontinuities $a_{\mu\nu}$, under local coordinates, transforms as

$$a'_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta (a_{\mu\nu} + t_{(\mu} l_{\nu)}), \tag{16}$$

where $l_\mu \equiv f_{,\mu}$.

The discontinuity of the Jacobian matrix J^μ_α is defined by the equation

$$\left[J^\mu_{\alpha,\beta,\gamma}\right]_\Sigma = t^\mu t_\alpha l_\beta l_\gamma, \tag{17}$$

where t_α is an arbitrary vector.

As a consequence, we find the following the discontinuity relations for the Riemann tensor:

$$[R_{\alpha\beta\mu\nu}]_\Sigma = \frac{1}{2}(a_{\alpha\mu} l_\beta l_\lambda + a_{\beta\lambda} l_\alpha l_\mu - a_{\alpha\lambda} l_\beta l_\mu - a_{\beta\mu} l_\alpha l_\lambda), \tag{18}$$

and for the Ricci tensor:

$$[R_{\alpha\beta}]_\Sigma = \frac{1}{2} g^{\rho\sigma}(a_{\alpha\rho} l_\beta l_\sigma + a_{\beta\sigma} l_\alpha l_\rho - a_{\rho\sigma} l_\beta l_\alpha - a_{\alpha\beta} l_\rho l_\sigma). \tag{19}$$

Einstein's equations for an empty space-time on Σ imply that $[R_{\mu\nu}]_\Sigma = 0$, if and only if the null vector l^α is an eigenvector of the matrix $a_{\mu\nu}$, that is, if and only if

$$a_{\alpha\beta} l^\beta = \frac{a}{2} l_\alpha, \tag{20}$$

where $a \equiv g^{\mu\nu} a_{\mu\nu}$.

Equation (20) shows that the coefficients of the discontinuity are not arbitrary. Let us analyze the discontinuity relation imposed by the Bianchi identity for the Riemann tensor. First of all, consider $f(x^\alpha)=0$ as the local equation of the hyper-surface Σ. From the definition of l_α, it has null vorticity, i.e., $l_{[\alpha;\beta]}=0$. The cyclic identity applied to the discontinuity of the Riemann tensor implies that

$$l_\rho [R_{\alpha\beta\lambda\mu}] + l_\lambda [R_{\alpha\beta\mu\rho}] + l_\mu [R_{\alpha\beta\rho\lambda}] = 0, \tag{21}$$

and

$$l_\rho [R^\rho{}_{\mu\alpha\beta}] = 0. \tag{22}$$

The covariant derivative of (21) yields

$$(l_\rho [R_{\alpha\beta\lambda\mu}] + l_\lambda [R_{\alpha\beta\mu\rho}] + l_\mu [R_{\alpha\beta\rho\lambda}])_{;\nu} = 0. \tag{23}$$

We contract the indices ρ and ν and then, considering that the Einstein equations for an empty space-time ($R_{\mu\nu}=0$) holds on Σ, obtain

$$2 l^\rho [R_{\alpha\beta\lambda\mu}]_{;\rho} + l^\rho{}_{;\rho} [R_{\alpha\beta\lambda\mu}] = 0. \tag{24}$$

Equation (24) tells us that if $[R_{\alpha\beta\lambda\mu}]$ vanishes at some point x of Σ, then it vanishes along the entire isotropic geodesic through x.

In the general case, for which l^α can either be space-type or null-like, we have the following result: let Ω be an oriented hyper-surface in the space-time intersecting $\Sigma(x^0 = 0)$ and defining a two-surface U. If one gives Cauchy's data $g_{\mu\nu}$ and $g_{\mu\nu,\lambda}$ on Ω, such that crossing on Σ the second derivatives admit the discontinuities $[g_{\alpha\beta,00}] = (a_{\alpha\beta})_\Sigma$, then $a_{\alpha\beta}$ on U must satisfy the condition

$$\left(a_{\mu\nu} - \frac{a}{2} g_{\mu\nu}\right) l^\nu \Big|_U = 0. \tag{25}$$

Equivalently, if for all points x of U one gives the tensor $[R_{\alpha\beta\mu\nu}]_U$ admitting as fundamental vector $(l^\mu)_U$, which is zero when contracted to the Riemann tensor, then the considered Cauchy data correspond to the solution of Einstein's equations such that the curvature tensor, when crossing Σ, admits a discontinuity $[R_{\alpha\beta\mu\nu}]$. The tensor $[R_{\alpha\beta\mu\nu}]$ is necessarily the solution of (24) corresponding to the initial data $[R_{\alpha\beta\mu\nu}]_U$.

The above results for non-empty Einstein's equations $G_{\mu\nu} = -kT_{\mu\nu}$ are easily proven, given the continuity of $T_{\mu\nu}$ through Σ and can be found in Lichnerowicz [88] or Novello and Salim [119]. From these considerations, we can draw the following lemma:

Lemma (**Lichnerowicz**) *Bianchi's identity together with the convenient Cauchy data represented by* (25) *are equivalent to Einstein's equations.*

This is the main result that will be used in this review. As an application of this method to deal with the quasi-Maxwellian formalism, we next analyze a few special solutions of the equations of general relativity.

2 Particular Solutions of GR from QM Equations

As specific examples of how the QM equations work, we reproduce certain known solutions of the general relativity theory using the quasi-Maxwellian framework. Our task is simplified in Gaussian coordinates, for which $g_{0\mu} = \delta^0_\mu$ and, moreover, a foliation described by the observer field $V^\mu \equiv \delta^\mu_0$. We use this coordinate system to deduce all the solutions in this section.

2.1 Schwarzschild Solution

In a Gaussian coordinate system, the Schwarzschild metric takes the form

$$ds^2 = dT^2 - B(T,R)dR^2 - r^2(T,R)d\Omega^2. \tag{26}$$

A geodesic observer in the metric (26) is $V^\mu = \delta^\mu_0$. The expansion coefficient θ is given by the expression

$$\theta = \frac{1}{2}\left(\frac{\dot{B}}{B} + \frac{4\dot{r}}{r}\right), \tag{27}$$

where $\dot{Y}(T,R) \equiv \partial Y/\partial T$.

Using the metric (26), we write the corresponding shear tensor $\sigma^\mu{}_\nu$ and the electric part of Weyl tensor $E^\mu{}_\nu$ in the matrix form

$$[\sigma^i{}_j] = f(T,R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{28}$$

and

$$[E^i{}_j] = g(T,R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{29}$$

where

$$f(T,R) = \frac{1}{3}\left(\frac{\dot{B}}{B} - \frac{2\dot{r}}{r}\right), \tag{30}$$

and

$$12g(T,R) = -2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - 4\frac{r''}{rB} + 2\frac{\dot{r}}{r}\frac{\dot{B}}{B} + 2\frac{r'}{r}\frac{B'}{B^2} + 4\frac{\ddot{r}}{r} - \frac{4}{r^2} - 4\frac{\dot{r}^2}{r^2} + 4\frac{r'^2}{r^2 B}. \tag{31}$$

Here, $Y'(T,R) \equiv \partial Y/\partial R$.

The magnetic part of Weyl tensor $H_{\alpha\beta}$, the vorticity $\omega_{\alpha\beta}$ and the acceleration a^μ are identically zero. The set of quasi-Maxwellian (11)–(14) reduces to the form

$$g' + 3\frac{r'}{r}g = 0, \tag{32a}$$

$$\dot{g} + 3\frac{\dot{r}}{r}g = 0 \tag{32b}$$

The evolution equations of the remaining kinematical quantities are provided by the Raychaudhuri equation and the shear evolution

$$\dot{\theta} + \frac{\theta^2}{3} + \frac{3}{2}f^2 = 0, \tag{33a}$$

$$\dot{f} + \frac{f^2}{2} + \frac{2}{3}\theta f = -g. \tag{33b}$$

Finally, the only nontrivial remaining constraint equation is

$$f' + 3\frac{r'}{r}f - \frac{2}{3}\theta' = 0. \tag{34}$$

This is but the Schwarzschild solution in Gaussian coordinates. Indeed, the functions $B(T,R)$ and $r(T,R)$ are obtained by imposing the Lichnerowicz condition

$$\left(R_{\mu\nu} = 0\right)\Big|_{T_0} \tag{35}$$

on the Cauchy surface $T = T_0$

It then follows that

$$B_E(T,R) = \frac{r'^2}{1+F(R)}, \tag{36a}$$

$$\dot{r}_E(T,R) = -\sqrt{F + r_H/r}, \tag{36b}$$

$$r'_E(T,R) = w'(R)\sqrt{F + r_H/r}, \tag{36c}$$

where $F(R)$ and $w(R)$ are arbitrary functions and r_H is an arbitrary constant.[1] These expressions play the role of initial conditions on the Cauchy surface, such that the functions B_{QM} and r_{QM} must be equal to B_E and r_E on T_0, and then, they are evolved to the entire space-time.

From (32a, 32b), it follows that

$$g = -\frac{k}{r^3}, \tag{37}$$

where k is another arbitrary constant.

From (34), we have that

$$B_{QM} = \frac{r'^2}{1+h(R)}, \tag{38}$$

where $h(R)$ is an arbitrary function.

One can write (33a, 33b) in terms of the functions B and r, as follows

$$\frac{\ddot{B}}{B} - \frac{1}{2}\frac{\dot{B}^2}{B^2} - 2\frac{\ddot{r}}{r} = -3g, \tag{39a}$$

$$\frac{\ddot{B}}{B} - \frac{1}{2}\frac{\dot{B}^2}{B^2} + 4\frac{\ddot{r}}{r} = 0. \tag{39b}$$

Substitution of (38) in (39b) yields

$$\frac{\ddot{r}'}{\ddot{r}} + 2\frac{r'}{r} = 0, \tag{40}$$

which can be integrated with respect to R. It results that

$$\ddot{r} = \frac{a(T)}{r^2}, \tag{41}$$

where $a(T)$ is an arbitrary function of T.

[1] We introduce the subscript indexes E and QM to distinguish the solution of the Einstein equations valid only on the Cauchy surface and the solution propagated by the quasi-Maxwellian equations, respectively.

Subtraction of (39a) from (39b) followed by manipulation of (38) and (37) shows that $a(T)$ is a constant ($a(T) = -k/2$). From (41), we find that

$$\dot{r}^2_{QM} = 2y(R) + \frac{k}{r}, \tag{42}$$

where $y(R)$ is an arbitrary function.

We then substitute the constraint (38) in the definition of g—(31)—and again subtract (39a, 39b), to find that

$$3\frac{\ddot{r}}{r} - \frac{\dot{r}}{r}\frac{\dot{r}'}{r'} + \frac{1}{2}\frac{h'}{rr'} + \frac{\dot{r}^2}{r^2} - \frac{h}{r^2} = 0. \tag{43}$$

Substituting (42) into (43) yields the relation

$$\frac{h}{r^2} - \frac{2y}{r^2} = x(T), \tag{44}$$

where $x(T)$ is an arbitrary function of T.

Equation (42) now yields the result

$$\int \frac{dr}{\sqrt{h + k/r}} = \int dT, \tag{45}$$

which we integrate to find that

$$\frac{\sqrt{(hr+k)r}}{h} - \frac{k}{2h^{3/2}} \ln\left(k + 2hr + 2\sqrt{(hr+k)hr}\right) = T + b(R). \tag{46}$$

Partial differentiation of (46) with respect to R yields the expression

$$r'_{QM} = b'\sqrt{h + \frac{k}{r}}. \tag{47}$$

We still have to determine the arbitrary functions in (36a, 36b and 36c). To this end, we set

$$F \equiv \alpha^2 - 1, w(R) \equiv \alpha R, \tag{48}$$

where α is a constant parameter.

We then consider the Jacobian matrix $J^\alpha{}_\beta$, given by the expression

$$[J^\alpha{}_\beta] \equiv \left[\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right] = \begin{pmatrix} \alpha/A & -(\alpha^2 - A)/A & 0 & 0 \\ -\sqrt{\alpha^2 - A} & \alpha\sqrt{\alpha^2 - A} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{49}$$

where $x^\alpha = (t, r, \theta, \phi)$, $\bar{x}^\beta = (T, R, \theta, \phi)$ and $A = 1 - r_H/r$, and map the metric given in Gaussian coordinates by (26) into the well-known Schwarzschild solution in the usual Schwarzschild coordinates

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{50}$$

if we make the identification $r_H = 2MG/c^2$.

The parameter α is interpreted as the test-particle mechanical energy resultant from integration of the geodesic equations.

Given the similarity between the functional forms of (38), (42) and (47), and of (36a, 36b and 36c), the arbitrary functions on the hyper-surface T_0 are trivially determined, as follows:

$$B_E(T_0, R) = B_{QM}(T_0, R) \Longrightarrow h(R) = F(R), \tag{51a}$$

$$\dot{r}_E(T_0, R) = \dot{r}_{QM}(T_0, R) \Longrightarrow k = k_1, \tag{51b}$$

$$r'_E(T_0, R) = r'_{QM}(T_0, R) \Longrightarrow b(R) = w(R). \tag{51c}$$

The metric (26) therefore becomes

$$ds^2 = dT^2 - \left(\alpha^2 - 1 + \frac{2M}{r(T,R)}\right) dR^2 - r^2(T,R) d\Omega^2. \tag{52}$$

It is more difficult to obtain the Schwarzschild internal solution along the above steps, because the Cauchy surface T_0 for this case is different from the spherically symmetric surface $r = r_0$—used in Schwarzschild coordinates to apply the matching conditions between the internal and the external parts. Besides, in contrast with the Schwarzschild observers $V^\mu = \sqrt{g_{00}}\delta^\mu_0$, Gaussian observers do not decompose the energy-momentum tensor as a perfect fluid, which makes calculations more cumbersome. Indeed, the energy-momentum tensor associated to the Gaussian observers δ^μ_0 is

$$T^{(G)}_{\mu\nu} = (\rho_G + p_G)V^\mu V^\nu - p_G g_{\mu\nu} + V_{(\mu}q_{\nu)} + \pi_{\mu\nu}. \tag{53}$$

The energy-momentum tensor, decomposed in terms of the observer field $u_\mu = e^{-\nu(T,R)}(\alpha, 1, 0, 0)$ is given by the expression

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu - p g_{\mu\nu}, \tag{54}$$

where $\nu = \nu(T, R)$ must satisfy the Tolman-Oppenheimer-Volkov equation and α is the external parameter associated with the co-moving test particle.

The physical properties of the fluid in the two representations are linked by the following expressions:

$$\rho_G = (\rho + p)\alpha^2 e^{-\nu} - p, \tag{55a}$$

$$p_G = -\frac{1}{3}[(\rho + p)(1 - \alpha^2 e^{-\nu}) - 3p], \tag{55b}$$

$$q^i = (\rho + p)\alpha e^{-\nu}\, \delta^i_1, \tag{55c}$$

$$\pi^i{}_j = \frac{2}{3}(1 - \alpha^2 e^{-\nu})\,\mathrm{diag}(1, -1/2, -1/2). \tag{55d}$$

Substituting these equations in the quasi-Maxwellian equations, one exactly finds the Schwarzschild internal solution containing some arbitrary functions. Matching conditions must be used to join this solution to Einstein's equations on the hyper-surface. Choosing the Cauchy surface $T = T_0$, one fixes all arbitrary functions, a procedure that yields the Schwarzschild stellar solution.

2.2 Kasner Solution

By contrast with the case of Schwarzschild metric, we shall use the Hadamard method to obtain the Kasner solution. We set for the Bianchi-I anisotropic metric the form

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \tag{56}$$

In this case, there is only one nontrivial QM-equation: the "time" evolution of the electric part of the Weyl tensor, which reads

$$\dot{E}^{\epsilon\lambda} + \theta\, E^{\epsilon\lambda} - \frac{3}{2}\,\sigma^{\mu(\epsilon}\, E^{\lambda)}{}_{\mu} + h^{\epsilon\lambda}\sigma^{\mu}{}_{\nu}\, E^{\nu}{}_{\mu} = 0, \tag{57}$$

where we have considered an observer field $V^{\mu} \equiv \delta^{\mu}_{0}$ and $T_{\mu\nu} = 0$.

The nontrivial equations for the kinematical quantities are

$$\dot{\theta} + \frac{\theta^2}{3} + 2\sigma^2 = 0, \tag{58}$$

and

$$\dot{\sigma}_{\alpha\beta} + E_{\alpha\beta} - \frac{2}{3}h_{\alpha\beta}\sigma^2 + \frac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta} = 0. \tag{59}$$

To obtain the Kasner solution, we set

$$a(t) = t^{p_1}, b(t) = t^{p_2}, c(t) = t^{p_3}, \tag{60}$$

where p_1, p_2 and p_3 must satisfy the equalities

$$\begin{aligned} &p_1 + p_2 + p_3 = 1, \\ &(p_1)^2 + (p_2)^2 + (p_3)^2 = 1. \end{aligned} \tag{61}$$

It is necessary to consider the Kasner solution on a Cauchy hyper-surface $t = t_0$ as initial condition for (57)–(59). One can then reduce the time evolution of $E_{\mu\nu}$, $\sigma\,\mu\nu$ and θ to a few algebraic constraints, that is, to

$$\begin{aligned} \dot{E}^{\mu}{}_{\nu} &= -2\theta E^{\mu}{}_{\nu}, \\ \dot{\sigma}^{\mu}{}_{\nu} &= -\theta\sigma^{\mu}{}_{\nu}, \\ \dot{\theta} &= -2\theta^2. \end{aligned} \tag{62}$$

Equation (62) are valid only on the Cauchy surface $t = t_0$. Using the relations (62) in (57), (58) and (59), these equations become just algebraic expressions for the Kasner background. As a consequence, one can define three special variables for the Kasner background:

$$\begin{aligned} X_{\alpha\beta} &\equiv E_{\alpha\beta} - \frac{2}{3}h_{\alpha\beta}\sigma^2 - \frac{1}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta}, \\ Y_{\alpha\beta} &\equiv \theta E_{\alpha\beta} + \frac{3}{2}\sigma^{\mu}{}_{(\alpha}E_{\beta)\mu} - h_{\alpha\beta}\sigma^{\mu}{}_{\nu}E^{\nu}{}_{\mu}, \\ W &\equiv 2\sigma^2 - \frac{2}{3}\theta^2. \end{aligned} \tag{63}$$

These quantities, which are identically zero for the Kasner solution (at t_0), represent the minimal set of variables which contains all the information about the metric because they come from the nontrivial equations of the QM-formalism. Once $X_{\alpha\beta}$, $Y_{\alpha\beta}$ and W are zero at t_0, the QM-equation will propagate them to the hyper-surface in the vicinity of t_0 retaining their null values due to their tensorial features. In other words, we show that the Kasner solution is valid for the entire space-time, according to the theorems of Section 1.3.

2.3 Friedman Solution

Consider the isotropic metric given in the Gaussian coordinate system:

$$ds^2 \equiv dt^2 + g_{ij}dx^i dx^j = dt^2 - a^2(t)\left[d\chi^2 + \sigma^2(\chi)d\Omega^2\right], \tag{64}$$

where $g_{ij} = -a^2(t)\gamma_{ij}(x^k)$.

The material content of this universe is represented by a perfect fluid, with energy density ρ, pressure p and equation of state $p = \lambda\rho$, where λ is a constant.

Inspection of (11)–(14) shows that the quasi-Maxwellian equations for this metric are identically zero because the metric is conformally flat (we use an observer field $V^{\mu} = \delta^{\mu}_{0}$). The only surviving kinematical equation is the Raychaudhuri equation

$$\dot{\theta} + \frac{1}{3}\theta^2 = -\frac{1}{2}(1 + 3\lambda)\rho. \tag{65}$$

The energy-momentum conservation reduces to the equalities

$$\dot{\rho} + (\rho + p)\theta = 0, a \tag{66a}$$

$$h_{\alpha}{}^{\mu} p_{,\mu} = 0, \Longrightarrow p = p(t). \tag{66b}$$

Therefore,

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(1 + 3\lambda)\rho, \tag{67}$$

and

$$\dot{\rho} + 3(1 + \lambda)\rho\frac{\dot{a}}{a} = 0. \tag{68}$$

Equation (68) can be integrated to yield the equality

$$\rho = \rho_0 a^{-3(1+\lambda)}, \tag{69}$$

where ρ_0 is a constant.

Substituting (69) in (65), we obtain the Friedman equation

$$\frac{\dot{a}^2}{a^2} - \frac{\epsilon}{a^2} = \frac{1}{3}\rho, \tag{70}$$

where ϵ is a constant of integration.

Finally, we can exhibit the scale factor $a(t)$ in terms of a quadrature equation

$$\int \frac{da}{\sqrt{\rho_0 a^{-(1+3\lambda)} + 3\epsilon}} = \frac{1}{\sqrt{3}}(t - t_0), \tag{71}$$

where t_0 is a constant of integration.

The initial conditions necessary to solve this problem are $a(t)$, $\dot{a}(t)$, $\ddot{a}(t)$ and $\sigma(r)$ on the Cauchy surface. On the other hand, we have λ, ϵ, ρ_0 and t_0. Instead of specifying each initial condition for the Cauchy problem, one can equivalently fix each free parameter. This can be done if we write the Riemann, Ricci and curvature tensors in terms of the three-geometry of the background $h_{\mu\nu}$, as:

$$\hat{R}_{\alpha\beta\mu\nu} = -\frac{\epsilon}{a^2}\,(h_{\alpha\mu}\,h_{\beta\nu} - h_{\alpha\nu}\,h_{\beta\mu}),$$
$$\hat{R}_{\beta\nu} = -\frac{2\epsilon}{a^2}\,h_{\beta\nu},$$
$$\hat{R} = -\frac{6\epsilon}{a^2},$$

where we have used the following relation, which holds for the 3-geometry:

$$\hat{\nabla}_\alpha\,\hat{\nabla}_\beta\,\hat{X}_\gamma - \hat{\nabla}_\beta\,\hat{\nabla}_\alpha\,\hat{X}_\gamma = -\hat{R}^{\lambda}_{\gamma\beta\alpha}\,\hat{X}_\lambda,$$

Here, the symbol ($\hat{}$) denotes projection on the hyper-surface defined by $h_{\mu\nu}$.

The explicit expression of $\hat{R}$ is obtained from the Friedman metric as follows

$$-6\epsilon \equiv \hat{R} - a^2 = -4\frac{\sigma''}{\sigma} + \frac{2}{\sigma^2} - 2\frac{\sigma'^2}{\sigma^2}. \tag{72}$$

The only three possible solutions for (72) are listed in Table 1, which joins the solutions of Friedmann equation for different values of λ and ϵ. The constant a_0, which is written in terms of ρ_0 and t_0, takes a different value for each solution in the table and it is commonly interpreted as the "current size of the Universe."

Table 1 Fundamental quantities of Friedman Universe (Units system k=c=1)

ρ	λ	ϵ	θ	$a(\eta)$	$t(\eta)$
$\frac{4}{3}t^{-2}$	0	0	$2t^{-1}$	$a_0\eta^2$	η^3
$\frac{3}{4}t^{-2}$	1/3	0	$\frac{3}{2}t^{-1}$	$a_0\eta$	η^2
$\frac{6}{a_0^2}(1-\cos\eta)^{-3}$	0	1	$\frac{3}{a_0}\frac{\sin\eta}{(1-\cos\eta)^2}$	$a_0(1-\cos\eta)$	$a_0(\eta-\sin\eta)$
$\frac{3}{a_0^2}\frac{1}{\sin^4\eta}$	1/3	1	$\frac{3}{a_0^2}\frac{\cos\eta}{\sin^2\eta}$	$a_0\sin\eta$	$a_0(1-\cos\eta)$
$\frac{6}{a_0^2}(\cosh\eta-1)^{-3}$	0	−1	$\frac{3}{a_0}\frac{\sinh\eta}{(\cosh\eta-1)^2}$	$a_0(\cosh\eta-1)$	$a_0(\sinh\eta-\eta)$
$\frac{3}{a_0^2}\frac{1}{\sinh^4\eta}$	1/3	−1	$\frac{3}{a_0^2}\frac{\cosh\eta}{\sinh^2\eta}$	$a_0\sinh\eta$	$a_0(\cosh\eta-1)$

2.4 Nonsingular Solutions

There are many proposals of cosmological solutions without a primordial singularity. The models are based on a variety of mechanisms, such as cosmological constant, nonminimal couplings, nonlinear Lagrangians involving quadratic terms in the curvature, modifications of the geometric structure of space-time and non-equilibrium thermodynamics, among others—cf. de Sitter [146], Murphy [104], Novello and Salim [117], Salim and de Olivera [142], Mukhanov and Brandenberger [101], Mukhanov and Sornborger [24], Novello et al. [116], Moessner and Trodden [99] and Saa et al. [144]. Recently, an inhomogeneous anisotropic nonsingular model for the early universe filled with a Born-Infield-type nonlinear electromagnetic field was presented by Garcia-Salcedo and Breton [54]. Additional investigations on regular cosmological solutions can be found in Klippert et al. [84], Veneziano [157] or Acácio de Barros et al. [2]. A complete listing of nonsingular solutions is presented in Novello and Bergliaffa [111]. Here, we shall analyze a few of the examples in the literature—cf. [30] and [39], for instance—using the quasi-Maxwellian formalism.

2.4.1 A WIST Model

In the Weyl integrable space-time model (WIST)—cf. Novello et al. [116], Salim and Sautu [143] and Fabris et al. [52]—as well as in string theory (Gasperini [56, 57]), there are models that describe the geometry $g_{\mu\nu}$ coupled to a scalar field. In those models, there are nonsingular solutions for an FLRW geometry. To search for a simple bounce scenario in cosmology, described by an analytical exact solution, we fix our attention on the background discussed by Novello et al. [116], Salim and Sautu [143], and Oliveira et al. [127]. Basically, the model concerns a modified Riemannian geometry with metric $g_{\mu\nu}$ and an extra Weyl affinity given by the expression

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {}^{\mu}_{\alpha\beta} \right\} + \frac{1}{2}\left(\delta^{\mu}_{\alpha}\omega_{,\beta} + \delta^{\mu}_{\beta}\omega_{,\alpha} - g_{\alpha\beta}\omega^{,\mu}\right).$$

In the Weyl manifold, the vacuum field equations can be rewritten in terms of a Riemannian geometry plus a term dependent of the Weyl field ω included in the nonmetric part of the affinity. At this level, the field equations can be represented by a perfect fluid with the four-velocity given by $V_\mu = \partial_\mu\omega/\omega^2$, where $\omega^2 \equiv g^{\alpha\beta}\omega_{,\alpha}\omega_{,\beta}$ and equation of state $p = \rho$. Originally, the scalar field is part of the affinity. However, it is transposed to the right-hand side of the field equations and can be interpreted as a perfect fluid. In this case, its effective energy density emerges as a negative quantity. The quasi-Maxwellian background equations of motion written in the conformal time are

$$(a')^2 + \epsilon a^2 + \frac{\lambda^2}{6}(w'a)^2 = 0, \tag{73}$$

and

$$w' = \gamma a^{-2}, \tag{74}$$

where γ is a constant and λ^2 is the coupling constant between the scalar field and the metric tensor.

It follows from (73) and (74) that

$$(a')^2 = -\epsilon a^2 - \frac{a_0^2}{a^2}, \tag{75}$$

where we have defined $a_0^2 \equiv \lambda\gamma/\sqrt{6}$.

Only solutions with three $\epsilon = -1$ curvatures are possible. The scale factor, solution to (73), is given by the equality

$$a(\eta) = a_o\sqrt{\cosh(2\eta + \delta)}, \tag{76}$$

where δ is a constant of integration.

The scalar factor displays a bounce produced by the scalar field that was introduced as the Weyl part of the affinity, due to the nonmetricity condition.

2.4.2 Nonsingular Solution from Nonlinear Electrodynamics

The standard cosmological model, based on the Friedman-Lemaître-Robertson-Walker (FLRW) geometry with Maxwell electrodynamics as its source, leads to a cosmological singularity at a finite time in the past as seen in Section 2.3. This mathematical singularity itself shows that, around the point of maximum condensation, the curvature and the energy density are arbitrarily large, hence beyond the domain of applicability of the model. This difficulty also raises secondary problems, such as the horizon problem: the Universe seems to be very homogeneous over scales approaching its causally correlated region, as pointed out by Brandenberger [22]. These secondary problems are usually solved by introducing geometric scalar fields (for a review on this approach see Kofman et al. [85] and references therein).

We now present the homogeneous and isotropic nonsingular FLRW solutions that are obtained by considering a toy model generalization of Maxwell electrodynamics—cf. Tsagas [154] for a review on electrodynamics in curved space-times. The model is presented as a local covariant, gauge-invariant Lagrangian dependent on the field invariants up to the second order, as a source of classical Einstein equations. This modification is expected to be relevant when the fields reach high values, as occurs in the primeval era of the Universe. Consequences of the inevitability of the singularity through the singularity theorems (see, Hawking and Ellis [64]) are circumvented by the appearance of a high (nevertheless finite) negative pressure in the early phase of the FLRW geometry. The influence of other kinds of matter on the evolution of the universe are also taken into account. It is shown that standard matter, even in its ultra-relativistic state, is unable to modify the regularity of the resulting solution.

Heaviside, nonrationalized units are used. The volumetric spatial average of an arbitrary quantity X at a given instant of time t is defined as

$$\langle X \rangle \equiv \lim_{V\to V_o} \frac{1}{V}\int \sqrt{-g}\, d^3x^i\, X, \tag{77}$$

where $V = \int \sqrt{-g}\, d^3x^i$ and V_o stands for the time-dependent volume of the whole space. An extended discussion of averages in cosmological models can be found in [15, 55, 162, 163].

Averaging Process Since the spatial sections of the FLRW geometry are isotropic, electromagnetic fields can generate such a universe only if an averaging procedure is performed—cf. Tolman and Ehrenfest [152], Hindmarsh and Everett [67]. The standard way to do this is simply to set the following mean values for the electric E_i and magnetic B_i fields:

$$\langle E_i \rangle = 0, \qquad \langle B_i \rangle = 0, \qquad \langle E_i B_j \rangle = 0, \tag{78}$$

$$\langle E_i E_j \rangle = -\frac{1}{3}E^2 g_{ij}, \tag{79}$$

and

$$\langle B_i B_j \rangle = -\frac{1}{3}B^2 g_{ij}. \tag{80}$$

The energy-momentum tensor associated with Maxwell Lagrangian is given by the equality

$$T_{\mu\nu} = F_{\mu}{}^{\alpha} F_{\alpha\nu} + \frac{1}{4} F g_{\mu\nu}, \tag{81}$$

where $F \equiv F_{\mu\nu} F^{\mu\nu} = 2\left(H^2 - E^2\right)$.

From the above average values, it follows that (81) reduces to a perfect fluid configuration with energy density ρ_γ and pressure p_γ as

$$\langle T_{\mu\nu} \rangle = (\rho_\gamma + p_\gamma) V_\mu V_\nu - p_\gamma g_{\mu\nu}, \tag{82}$$

where

$$\rho_\gamma = 3 p_\gamma = \frac{1}{2}\left(E^2 + B^2\right). \tag{83}$$

From the Raychaudhuri equation, we can see that the singular nature of the FLRW universes comes from the positive definiteness at all times of both the energy density and pressure. The Einstein equations for the above energy-momentum configuration therefore yield

$$a(t) = \sqrt{a_o^2 t - \epsilon t^2}, \tag{84}$$

where a_o is an arbitrary constant.

Nonsingular FLRW Universes Nonlinear generalization of Maxwell electromagnetic Lagrangian will be considered up to second-order terms in the field invariants F and $G \equiv \frac{1}{2}\eta_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu} = -4(\vec{E} \cdot \vec{B})$ as

$$L = -\frac{1}{4} F + \alpha F^2 + \beta G^2, \tag{85}$$

where α and β are arbitrary constants.[2] Maxwell electrodynamics can be formally obtained from (85) by setting $\alpha = \beta = 0$. Alternatively, it can also be dynamically obtained from the nonlinear theory in the limit of small fields. The energy-momentum tensor for arbitrary nonlinear electromagnetic theories reads

$$T_{\mu\nu} = -4 L_F F_{\mu}{}^{\alpha} F_{\alpha\nu} + (G L_G - L) g_{\mu\nu}, \tag{86}$$

where L_F represents the partial derivative of the Lagrangian with respect to the invariant F and similarly for the invariant G. In the linear case, (86) reduces to the usual form (81).

Since we are mainly interested in the analysis of the early-universe behavior of this system, where matter should be identified with a primordial plasma—for instance, Tajima et al. [150], Giovannini and Shaposhnikov [58] and Campos and Hu [31]—we are led to limit our considerations to the case in which only the average of the squared magnetic field B^2 survives, as it was done by Tajima et al. [150], Dunne [43], Joyce and Shaposhnikov [81], Giovannini and Shaposhnikov [58] and Dunne and Hall [42]. This is formally equivalent to setting $E^2 = 0$ in (79). Physically, it amounts to neglecting the bulk-viscosity terms in the electric conductivity of the primordial plasma.

The homogeneous Lagrangian (85) requires some spatial averaging over large scales, as given by (78)–(80). If one intends to make similar calculations on smaller scales, then either more involved nonhomogeneous Lagrangians should be used or some additional magneto-hydrodynamical effect introduced, as was done by Thompson and Blaes [151] and Subramanian and Barrow [149], to achieve correlation at the desired scale (see Jedamzik et al. [76]). Since the averaging procedure is independent from the equations of the electromagnetic field, we can use the above (78)–(80) to come to a counterpart of (82) for the non-Maxwellian case. The average energy-momentum tensor is identified with the perfect fluid (82) with the following modified expressions for the energy density ρ_γ and pressure p_γ:

$$\rho_\gamma = \frac{1}{2} B^2 \left(1 - 8\alpha B^2\right), \tag{87}$$

and

$$p_\gamma = \frac{1}{6} B^2 \left(1 - 40\alpha B^2\right). \tag{88}$$

Insertion of (87) and (88) in the continuity equation (66a) for a Friedman model yields the relation

$$B = \frac{B_o}{a^2}, \tag{89}$$

where B_o is a constant.

With this result, a similar procedure applied to (70) leads to the expression

$$\dot{a}^2 = \frac{k B_o^2}{6a^2}\left(1 - \frac{8\alpha B_o^2}{a^4}\right) - \epsilon, \tag{90}$$

where k is the Einstein constant.

Since the right-hand side of (90) must not be negative, it follows that, regardless of ϵ, for $\alpha > 0$, the scale factor $a(t)$ cannot be arbitrarily small.

The solution of (90) is implicitly given by the equality

$$ct = \pm \int_{a_o}^{a(t)} \frac{dz}{\sqrt{\frac{kB_o^2}{6z^2} - \frac{8\alpha k B_o^4}{6z^6} - \epsilon}}, \tag{91}$$

[2] If we consider that the origin of these corrections comes from quantum fluctuations, then the value of the constants α and β are fixed—see Heisenberg and Euler [66].

where $a(0) = a_o$. To recover the linear case (84) from (91), we have to set $\alpha = 0$.

From (91), for $\epsilon = \pm 1$, the following closed form can be derived:

$$ct = \pm \left[\frac{(x_1 - x_3)\mathcal{E}\left(\arcsin\sqrt{\frac{z-x_1}{x_2-x_1}}, \sqrt{\frac{x_1-x_2}{x_1-x_3}}\right) + x_3\mathcal{F}\left(\arcsin\sqrt{\frac{z-x_1}{x_2-x_1}}, \sqrt{\frac{x_1-x_2}{x_1-x_3}}\right)}{\sqrt{x_1 - x_3}} \right]\Bigg|_{z=a_o^2}^{z=a^2(t)}, \tag{92}$$

where x_1, x_2 and x_3 are the three roots of the equation $8\alpha k B_o^4 - k B_o^2 x + 3\epsilon x^3 = 0$ and

$$\mathcal{F}(x, \kappa) \equiv \int_0^{\sin x} \frac{dz}{\sqrt{(1 - z^2)(1 - \kappa^2 z^2)}},$$
$$\mathcal{E}(x, \kappa) \equiv \int_0^{\sin x} \sqrt{\frac{1 - \kappa^2 z^2}{1 - z^2}} dz, \tag{93}$$

are the elliptic functions of the first and second kinds, respectively, (see expressions 8.111.2 and 8.111.3 in Gradshteyn and Ryzhik, 1965). Figure 1 shows $a(t)$ for $\epsilon = \pm 1$.

For the Euclidean section, suitable choice of the time origin yields the following solution to (91):

$$a^2 = B_o \sqrt{\frac{2}{3}(kc^2 t^2 + 12\alpha)}, \tag{94}$$

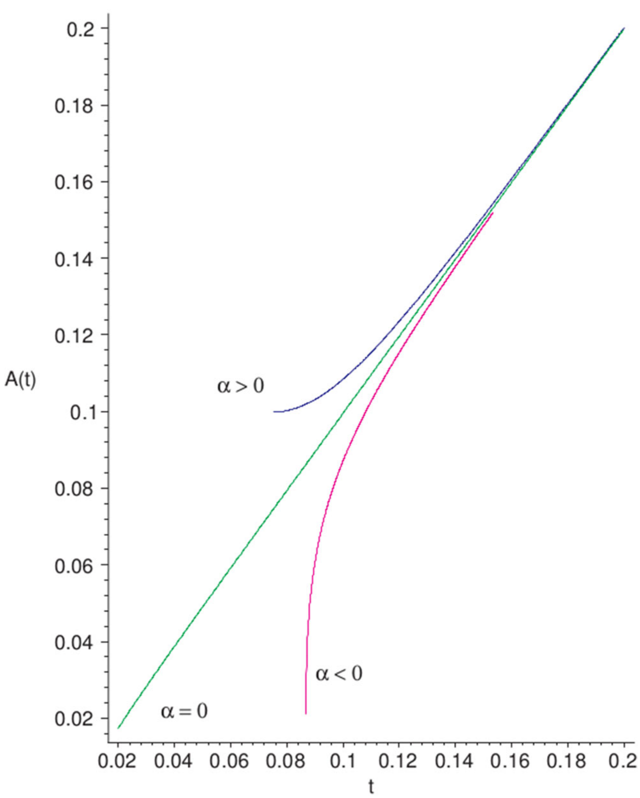

Fig. 1 Function $a(t)$ resulting from (91) for the illustrative choices $a(1) = 1$, $kB_o^2 = 12$ and $\alpha B_o^2 = (0; \pm 1, 25.10^{-4})$

and (89) yields the following expression for the time evolution of the average-strength magnetic field B:

$$B^2 = \frac{3}{2} \frac{1}{kc^2 t^2 + 12\alpha}. \tag{95}$$

Equation (94) is singular for $\alpha < 0$, as $a(t)$ becomes arbitrarily small at the time $t = \sqrt{-12\alpha/kc^2}$ for which . For $\alpha > 0$, at $t = 0$, the radius of the universe attains a minimum a_{min}, given by the expression

$$a_{min}^2 = B_o \sqrt{8\alpha}. \tag{96}$$

The minimum radius a_{min} depends on B_o, which turns out to be the only free parameter of the present model. The energy density ρ_γ given by (87) reaches its maximum $\rho_{max} = 1/64\alpha$ at the instant $t = t_c$, where

$$t_c = \frac{1}{c}\sqrt{\frac{12\alpha}{k}}. \tag{97}$$

For smaller t, the energy density decreases, vanishing at $t = 0$, while the pressure becomes negative. Only for times $t \lesssim 10\sqrt{\alpha/kc^2}$ are the nonlinear effects relevant for the cosmological solution of the normalized scale factor, as shown by Fig. 2. Indeed, the solution (94) fits the standard expression (84) for the Maxwell case at the limit of large times.

For $\alpha \neq 0$, the energy-momentum tensor (86) is not trace-free. Thus, the equation of state $p_\gamma = p_\gamma(\rho_\gamma)$ is no longer given by the Maxwellian value. It contains, instead, a quintessential-like term—see Caldwell et al. [28]—proportional to the constant α, that is,

$$p_\gamma = \frac{1}{3}\rho_\gamma - \frac{16}{3}\alpha B^4. \tag{98}$$

Equation (98) can also be written in the form

$$p_\gamma = \frac{1}{3}\rho_\gamma - \frac{1}{24\alpha}\Big\{(1 - 32\alpha\rho_\gamma) + [1 - 2\Theta(t - t_c)]\sqrt{1 - 64\alpha\rho_\gamma}\Big\}, \tag{99}$$

where $\Theta(z)$ is the Heaviside step function.

The right-hand side of (99) behaves as $(1 - 64\alpha\rho_\gamma)\rho_\gamma/3$ for $t > t_c$ in the Maxwell limit $\alpha\rho_\gamma \ll 1$.

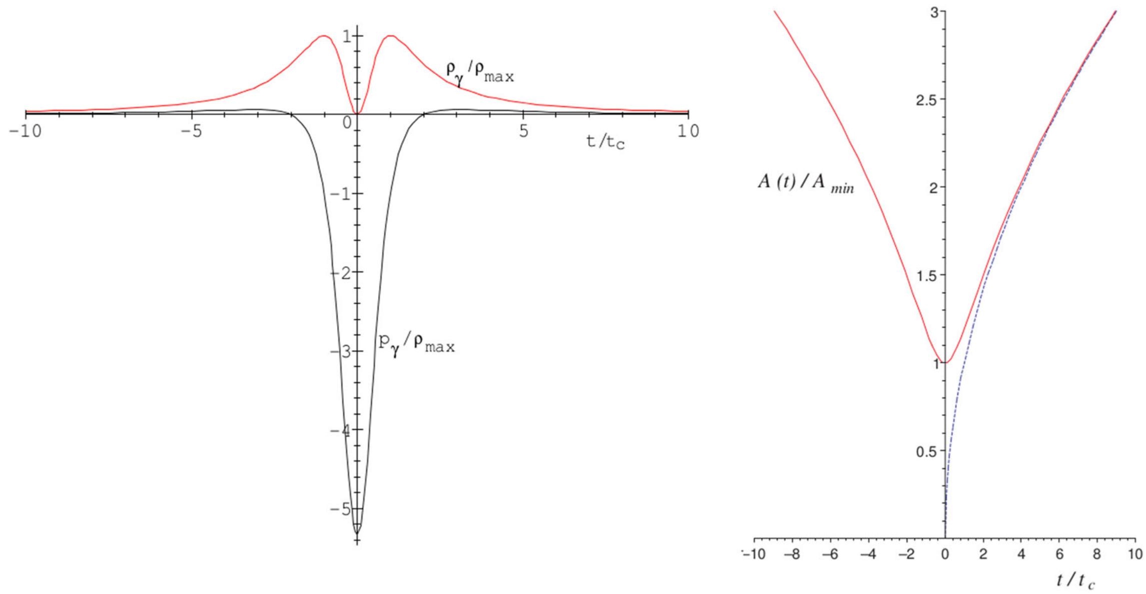

Fig. 2 *Left panel* time dependence of the electromagnetic energy density ρ_γ and pressure p_γ with $\rho_{max} = 1/64\alpha$ and t_c given by (97). *Right panel* nonsingular behavior of the scale factor $a(t)$. The dividing factors a_{min} and t_c are given by (96) and (97), respectively. For comparison, the *dashed line* shows the corresponding classical expression (84) with $a_o = a_{min}$

The maximum temperature corresponding to $t = t_c$ is given by

$$T_{max} = \left(\frac{c}{24\alpha\sigma}\right)^{1/4}, \tag{100}$$

where σ is the Stefan-Boltzmann constant.

From the cosmological point of view, therefore, only in the primeval era of the universe do the consequences of the minimal coupling of gravity with second-order nonlinear electrodynamics propose relevant modifications. Indeed, the class of $\alpha > 0$ theories leads to nonsingular solutions for which the scale factor $a(t)$ attains a minimum. The regularity of this cosmological solution is due to the quantity $\rho + 3p$ becoming negative for a certain interval of time.

3 Perturbation Theory in the QM Formalism

Since the original paper of Lifshitz and Khalatnikov [89], it has been common practice to consider variations of such nonobservable quantities as $\delta g_{\mu\nu}$ at the outset of the perturbation theory of Einstein's equations of general relativity. The main drawback of this procedure is mixing true perturbations with arbitrary coordinate transformations. A solution was found by looking for gauge-independent combinations, which have been written in terms of the metric tensor and its derivatives by many authors (cf. Hawking [65], Jones [77], Olson [128], Bardeen [9], Brandenberger [23], Vishniac [73] and Mukhanov [101]). Nowadays, this gauge-invariant approach can be easily compared with observational data, as detailed by Tsagas et al. [155].

The fundamental element of the gauge problem in RG perturbation theory was clearly, geometrically detailed by Stewart's Lemma [147, 148]: gauge invariant variables (scalars or not)[3] are those which are identically null on the background. After that, in Stewart's sense, Hawking [65] used the QM equations to argue that the applicability of this alternative RG formalism is restricted to the standard cosmology problem—the problem of a homogenous, isotropic and conformally flat case.

Although strictly correct, this argument left the QM formalism in disadvantage relative to the other methods based on the Lifshitz program and justified the wide use of the complex Newmann-Penrose formalism [106]. Nonetheless, we can prove that some objects of this formalism are physically unobservable.

Here, we will follow a simpler and more direct path, which corresponds to choosing, from the beginning, as the basis of our analysis, the gauge-invariant physically observable quantities. The dynamics for these fundamental quantities will then be analyzed and any remaining gauge-dependent objects that can be dealt with will be obtained from the fundamental set.

[3] We shall refer to the gauge-invariant (gauge-dependent) variables as "good" ("bad") ones, a terminology inspired in the Stewart's Lemma [148].

There are basically two fundamental approaches along which the perturbation theory can be developed: one of them makes use of Einstein's standard equations [89] and the other is based on the equivalent quasi-Maxwellian description (cf. Jordan [78], Hawking [65] and Novello and Salim [118]). In this paper, we will focus on the second approach.

3.1 Perturbed Quasi-Maxwellian Equations

We state here the perturbed linearized quasi-Maxwellian equations for gravity, which shall be used in the following sections to treat the dynamics of the perturbed quantities. We write all the dynamical variables in the form

$$A_{(perturbed)} = A_{(background)} + (\delta A).$$

Straightforward manipulations yield the following perturbed QM equations:

$$\begin{aligned} h_\mu{}^\alpha h_\nu{}^\beta (\delta E^{\mu\nu})^\bullet \;& +\theta(\delta E^{\alpha\beta}) - \frac{1}{2}(\delta E_\nu{}^{(\alpha})h^{\beta)}{}_\mu V^{\mu;\nu} \\ & +\frac{\theta}{3}\eta^{\beta\nu\mu\varepsilon}\eta^{\alpha\gamma\tau\lambda}V_\mu V_\tau(\delta E_{\varepsilon\lambda})h_{\gamma\nu} + \\ & -\frac{1}{2}(\delta H_\lambda{}^\mu)_{;\gamma}h_\mu{}^{(\alpha}\eta^{\beta-)\tau\gamma\lambda}V_\tau \\ = & -\frac{1}{2}(\rho+p)(\delta\sigma^{\alpha\beta}) + -\frac{1}{6}h^{\alpha\beta}(\delta q^\mu)_{;\mu} \\ & -\frac{1}{4}h^{\mu(\alpha}h^{\beta)\nu}(\delta q_\mu)_{;\nu} + \frac{1}{2}h^{\mu\alpha}h^{\beta\nu}(\delta\pi_{\mu\nu})^\bullet \\ & +\frac{1}{6}\theta\left(\delta\pi^{\alpha\beta}\right), \end{aligned} \tag{101}$$

$$\begin{aligned} h_\mu{}^\alpha h_\nu{}^\beta (\delta H^{\mu\nu})^\bullet \;& +\theta\left(\delta H^{\alpha\beta}\right) - \frac{1}{2}\left(\delta H_\nu{}^{(\alpha}\right)h^{\beta)}{}_\mu V^{\mu;\nu} \\ & +\frac{\theta}{3}\eta^{\beta\nu\mu\varepsilon}\eta^{\alpha\lambda\tau\gamma}V_\mu V_\tau(\delta H_{\varepsilon\gamma})h_{\lambda\nu} + \\ & -\frac{1}{2}\left(\delta E_\lambda{}^\mu\right)_{;\tau}h_\mu{}^{(\alpha}\eta^{\beta-)\tau\gamma\lambda}V_\gamma \\ & = \frac{1}{4}h^{\nu\alpha}\eta^{\beta\varepsilon\tau\mu}V_\mu(\delta\pi_{\nu\varepsilon})_{;\tau}, \end{aligned} \tag{102}$$

$$(\delta H_{\alpha\mu})_{;\nu}h^{\alpha\varepsilon}h^{\mu\nu} = (\rho+p)(\delta\omega^\varepsilon) - \frac{1}{2}\eta^{\varepsilon\alpha\beta\mu}V_\mu(\delta q_{up\alpha})_{;\beta}, \tag{103}$$

and

$$\begin{aligned}(\delta E_{\alpha\mu})_{;\nu}h^{\alpha\varepsilon}h^{\mu\nu} = & \frac{1}{3}(\delta\rho)_{,\alpha}h^{\alpha\varepsilon} - \frac{1}{3}\dot\rho(\delta V^\varepsilon) - \frac{1}{3}\rho_{,0}(\delta V^0)V^\varepsilon \\ & +\frac{1}{2}h^\varepsilon{}_\alpha(\delta\pi^{\alpha\mu})_{;\mu} + \frac{\theta}{3}(\delta q^\varepsilon). \end{aligned} \tag{104}$$

The perturbed equations for the kinematical quantities are

$$(\delta\theta)^\bullet + \dot\theta(\delta V^0) + \frac{2}{3}\theta(\delta\theta) - (\delta a^\alpha)_{;\alpha} = -\frac{(1+3\lambda)}{2}(\delta\rho), \tag{105}$$

$$\begin{aligned}(\delta\sigma_{\mu\nu})^\bullet \;& +\frac{1}{3}h_{\mu\nu}(\delta a^\alpha)_{;\alpha} - \frac{1}{2}(\delta a_{(\alpha})_{;\beta)}h_\mu{}^\alpha h_\nu{}^\beta \\ & +\frac{2}{3}\theta(\delta\sigma_{\mu\nu}) = -(\delta E_{\mu\nu}) - \frac{1}{2}(\delta\pi_{\mu\nu}), \end{aligned} \tag{106}$$

$$(\delta\omega^\mu)^\bullet + \frac{2}{3}\theta(\delta\omega^\mu) = \frac{1}{2}\eta^{\alpha\mu\beta\gamma}(\delta a_\beta)_{;\gamma}V_\alpha, \tag{107}$$

$$\begin{aligned}\frac{2}{3}(\delta\theta)_{,\lambda}h^\lambda{}_\mu \;& - \frac{2}{3}\dot\theta(\delta V_\mu) + \frac{2}{3}\dot\theta(\delta V^0)\delta_\mu{}^0 \\ & - (\delta\sigma^\alpha{}_\beta + \delta\omega^\alpha{}_\beta)_{;\alpha}h^\beta{}_\mu = -(\delta q_\mu), \end{aligned} \tag{108}$$

$$(\delta\omega^\alpha)_{;\alpha} = 0 \tag{109}$$

and

$$(\delta H_{\mu\nu}) = -\frac{1}{2}h^\alpha{}_{(\mu}h^\beta{}_{\nu)}((\delta\sigma_{\alpha\gamma})_{;\lambda} + (\delta\omega_{\alpha\gamma})_{;\lambda})\eta_{up\beta}{}^{\varepsilon\gamma\lambda}V_\varepsilon. \tag{110}$$

The perturbed equations for the conservation of the energy-momentum tensor yields the relations

$$(\delta\rho)^\bullet + \dot\rho(\delta V^0) + \theta(\delta\rho + \delta p) + (\rho+p)(\delta\theta) + (\delta q^\alpha)_{;\alpha} = 0, \tag{111}$$

and

$$\begin{aligned}\dot p(\delta V_\mu) + p_{,0}(\delta V^0)\delta_\mu{}^0 - (\delta p)_{,\beta}h^\beta{}_\mu + (\rho+p)(\delta a_\mu) \\ + h_{\mu\alpha}(\delta q^\alpha)^\bullet + \frac{4}{3}\theta(\delta q_\mu) + h_{\mu\alpha}\left(\delta\pi^{\alpha\beta}\right)_{;\beta} = 0. \end{aligned} \tag{112}$$

We will next present a few examples of how this perturbation method works. To this end, we consider the solutions derived in the last section.

3.2 Schwarzschild Solution

To analyze the stability of the Schwarzschild geometry, Regge and Wheeler [135] used the standard Lifshitz method. They discussed the spectral decomposition of linear perturbations in terms of two fundamental modes, called "even" and "odd," and concluded affirmatively on the stability of this geometry. The main difficulties identified in their work were solved by Vishveshwara [158] by means of a convenient coordinate transformation.

An important characteristic of these works, which has been insufficiently emphasized in the literature, is the impossibility of applying harmonic decomposition. In effect, the "even" and "odd" modes are both obtained from the scalar derivation, where only the degrees of freedom of spin 1 are present. Klippert [83] has shown that it is possible to develop a systematic way to appropriately apply the QM formalism to the Schwarzschild solution, even in Stewart's sense—cf. Stewart [148]. To review briefly this approach, we construct a *manifold basis*.

3.2.1 Construction of the Schwarzschild Basis

Consider the complete set of eigenfunctions of the Laplace-Beltrami operator ($\hat{\nabla}^2$). The manifold is a submanifold orthogonal to $V^\mu = \delta^\mu_0$ and we deal with the Schwarzschild geometry in Gaussian coordinates. The Laplacian is formally written in the form

$$\hat{\nabla}^2 \equiv \hat{\nabla}^\alpha \hat{\nabla}_\alpha$$

where $\hat{\nabla}_\alpha \equiv h_\alpha{}^\beta \nabla_\beta$ and $h_\alpha{}^\beta = \delta_\alpha{}^\beta - V_\alpha V^\beta$.

The scalar component of the Schwarzschild base is a function $Q(x^\alpha)$ such that

$$\hat{\nabla}^2 Q = -\lambda_s Q \tag{113}$$

explicitly in terms of the metric (52) yields

$$\frac{1}{r^2\sqrt{\alpha^2 - A}}\frac{\partial}{\partial R}\left(\frac{r^2}{\sqrt{\alpha^2 - A}}\frac{\partial Q}{\partial R}\right) + \frac{1}{r^2}\left[\frac{\partial^2 Q}{\partial \theta^2} + \cot\theta\frac{\partial Q}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial \phi^2}\right] = \lambda Q, \tag{114}$$

where $A(T, R) = 1 - 2M/r(T, R)$.

The angular variables will give origin to the spherical harmonics. Afterwards, we consider the particular case $\alpha^2 = 1$ to get the easier differential equation below for the basis Q in terms of T and R coordinates

$$\frac{3}{2}(T + R)\frac{\partial^2 F}{\partial R^2} + \frac{5}{2}\frac{\partial F}{\partial R} + \left[\frac{4}{3}\frac{l(l+1)}{T+R} + \lambda\sqrt{2M}\left(-\frac{3}{2}\sqrt{2M}(T+R)\right)^{1/3}\right]F = 0, \tag{115}$$

where it was assumed that $Q(T, R, \theta, \phi) \equiv F(T, R)Y^l_m(\theta, \phi)$, the $Y^l_m(\theta, \phi)$ being the spherical harmonics.

We can integrate (115) to obtain the general solution in terms of Bessel functions, as follows:

$$F(T, R) = \frac{1}{(T + R)^{1/3}}\left[a\, J_\alpha(x) + b\, Y_\alpha(x)\right], \tag{116}$$

where we have defined

$$\alpha \equiv \frac{\sqrt{1 - 8l - 8l^2}}{2}, \qquad x \equiv \frac{\sqrt{6c}\,(T + R)^{2/3}}{2}.$$

Here, $c \equiv -\lambda\sqrt{2M}[(3/2)\sqrt{2M}]^{1/3}$ and a and b are integration constants. $J_\alpha(x)$ and $Y_\alpha(x)$ are the Bessel functions of the first and second kind, respectively.

3.2.2 Gauge-Invariant Variables

Section 2.1 showed that the decomposition induced by $V^\mu = \delta^\mu_0$ leads to a degenerate shear tensor, with two identical eigenvalues proportional to the electric part of the Weyl tensor. We therefore introduce the following geometrical objects

$$\begin{aligned} X_{\mu\nu} &\equiv \sigma_{\mu\nu} - \tfrac{2\sigma^2}{\sigma^3}\sigma^\alpha{}_\mu\sigma_{\alpha\nu} + \tfrac{2\sigma^2}{\sigma^3}\tfrac{2\sigma^2}{3}h_{\mu\nu}, \\ Y_{\mu\nu} &\equiv E_{\mu\nu} - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\sigma_{\mu\nu}, \\ Z_{\mu\nu} &\equiv H_{\mu\nu} - \tfrac{H^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\sigma_{\mu\nu}. \end{aligned} \tag{117}$$

These tensors present special algebraic features: they are symmetric, traceless, orthogonal to the shear ($X^\alpha{}_\beta\sigma^\beta{}_\alpha = Y^\alpha{}_\beta\sigma^\beta{}_\alpha = Z^\alpha{}_\beta\sigma^\beta{}_\alpha = 0$) and, most importantly, null on the background. They therefore constitute a set of "good" variables to develop perturbation theory for the Schwarzschild case.

3.2.3 Dynamics

Using the QM-equations, we can calculate the propagation equations of $X_{\mu\nu}$, $Y_{\mu\nu}$ and $Z_{\mu\nu}$ along the geodesics represented by the vector field V^μ. It is useful to rewrite the outcome in terms of these objects to obtain a closed dynamical system. We restrict ourselves to the exhibition of the propagation equations for the perturbations associated to the gauge-invariant variables as follows:

$$\begin{aligned} \delta\dot{X}_{\mu\nu} = &-\left(\frac{4}{3}\theta + \frac{\sigma^3}{2\sigma^2} + 2\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\right)\delta X_{\mu\nu} - 2\sigma^\lambda{}_{(\mu}\delta X_{\nu)\lambda} \\ &+ \delta Y_{\mu\nu} + 2\frac{2\sigma^2}{\sigma^3}\sigma^\lambda{}_{(\mu}\delta Y_{\nu)\lambda} + \\ &+ \left[h^\alpha_\mu h^\beta_\nu - 2\frac{2\sigma^2}{\sigma^3}h^\alpha{}_{(\mu}\sigma_{\nu)}{}^\beta + \frac{1}{2\sigma^2}\left(\sigma^{\alpha\beta} - X^{\alpha\beta}\right)\right. \\ &\left.\times\left(\sigma_{\mu\nu} - X_{\mu\nu} + 2\frac{2\sigma^2}{\sigma^3}\frac{2\sigma^2}{3}h_{\mu\nu}\right)\right] \cdot \\ &\cdot\left(\delta a_{(\alpha;\beta)} + \frac{1}{2}\delta\pi_{\alpha\beta}\right) + \frac{1}{3}\delta a^\lambda{}_{;\lambda}\left[2\frac{2\sigma^2}{\sigma^3}\sigma_{\mu\nu} - h_{\mu\nu}\right], \end{aligned} \tag{118}$$

$$\delta\dot{Y}_{\mu\nu} = -4\frac{\sigma^3}{2\sigma^2}\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}X_{\mu\nu} + \left(\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2} - \theta\right)\delta Y_{\mu\nu} + 3\sigma^\lambda{}_{(\mu}\delta Y_{\nu)\lambda} + \left(h^\alpha_\mu h^\beta_\nu - \frac{1}{2\sigma^2}\sigma^{\alpha\beta}\sigma_{\mu\nu}\right) \cdot \cdot\left[h^\lambda{}_{(\alpha}\eta_{\beta)}{}^{\epsilon\gamma\tau}V_\tau\delta Z_{\lambda\epsilon;\gamma} - \frac{1}{2}\delta\dot{\Pi}_{\alpha\beta} + \left(\frac{\theta}{3} - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\right)\delta\Pi_{\alpha\beta} + \sigma^\lambda{}_{(\alpha}\delta\Pi_{\beta)\lambda} - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\delta a_{(\alpha;\beta)}\right] + - \sigma^\lambda{}_{(\mu}\delta\omega_{\nu)\lambda}\frac{1}{6}\left(\delta q^\lambda_{;\lambda} + \sigma^{\alpha\beta}\delta\Pi_{\alpha\beta} + 2\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\delta a^\lambda_{;\lambda}\right)h_{\mu\nu}, \tag{119}$$

and

$$\delta\dot{Z}_{\mu\nu} = \left(h^\alpha_\mu h^\beta_\nu - \frac{\sigma^{\alpha\beta}}{2\sigma^2}\sigma_{\mu\nu}\right)\Bigg\{h^\lambda{}_{(\alpha}\eta_{\beta)}{}^{\epsilon\gamma\tau}V_\tau \left(\delta Y_{\lambda\epsilon;\gamma} - \frac{1}{2}\delta\pi_{\lambda\epsilon;\gamma} - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\delta\omega_{\lambda\epsilon;\gamma}\right) + - \sigma^\lambda{}_{(\alpha}\eta_{\beta)\lambda\epsilon\gamma}V^\epsilon\left(2\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\delta a^\gamma - \frac{1}{2}\delta q^\gamma\right) - \delta\left[\sigma_{\lambda(\alpha}\eta_{\beta)}{}^{\lambda\gamma\tau}V_\tau\left(\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\right)_{;\gamma}\right]\Bigg\} + - \left(\theta - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\right)\delta Z_{\mu\nu} + 3\sigma^\lambda{}_{(\mu}\delta Z_{\nu)\lambda}, \tag{120}$$

Equations (118), (119) and (120) are not completely independent. They must satisfy constraint equations, which rewritten in terms of the above geometrical objects have the forms

$$h^\alpha_\epsilon h^\nu_\mu X^\mu{}_{\alpha;\nu} = -\frac{\sigma^3}{2\sigma^2}h^\alpha{}_\epsilon\sigma^{\mu\nu}\sigma_{\alpha\mu;\nu} + \left[\sigma^\alpha{}_\epsilon - X^\alpha{}_\epsilon + \frac{2\sigma^2}{\sigma^3}\frac{2\sigma^2}{3}h^\alpha{}_\epsilon\right] \times\frac{(\sigma^3/2\sigma^2)_{;\alpha}}{(\sigma^3/2\sigma^2)} + h^\alpha{}_\epsilon\left(\frac{2\sigma^2}{\sigma^3}\frac{2\sigma^2}{3}\right)_{;\alpha} + \left(h^\alpha_\epsilon - 2\frac{2\sigma^2}{\sigma^3}\sigma^\alpha{}_\epsilon\right)\left[h^{\mu\nu}\omega_{\alpha\mu;\nu} + \frac{2}{3}\theta_{;\alpha} + (\sigma_{\alpha\lambda} + \omega_{\alpha\lambda})a^\lambda - q_\alpha\right], \tag{121}$$

$$h^\alpha_\epsilon h^\nu_\mu Y^\mu{}_{\alpha;\nu} = \eta_\epsilon{}^{\alpha\beta\gamma}V_\gamma\sigma^\lambda{}_\alpha Z_{\beta\lambda} - 3\left(Z_{\epsilon\lambda} + \frac{H^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\sigma_{\epsilon\lambda}\right)\omega^\lambda - \sigma^\lambda{}_\epsilon\left(\frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2}\right)_{;\lambda} + \frac{1}{3}h^\alpha_\epsilon\rho_{;\alpha} + - \frac{E^\alpha{}_\beta\sigma^\beta{}_\alpha}{2\sigma^2} \left[h^\alpha_\epsilon h^{\mu\nu}\omega_{\alpha\mu;\nu} + \frac{2}{3}h^\alpha_\epsilon\theta_{;\alpha} + (\sigma_{\epsilon\lambda} + \omega_{\epsilon\lambda})a^\lambda - q_\epsilon\right] + \frac{1}{2}\left[(\sigma_{\epsilon\lambda} - 3\omega_{\epsilon\lambda})q^\lambda - h_{\alpha\epsilon}\pi^{\alpha\beta}{}_{;\beta} + \pi_{\epsilon\lambda}a^\lambda\right] - \frac{\theta}{3}q_\epsilon, \tag{122}$$

and

$$h^\alpha_\epsilon h^\nu_\mu Z^\mu{}_{\alpha;\nu} = \eta_\epsilon{}^{\alpha\beta\gamma}V_\gamma\sigma^\lambda{}_\alpha Y_{\beta\lambda} + 3\left(Y_{\epsilon\lambda} + \frac{\sigma_{\alpha\beta}E^{\alpha\beta}}{2\sigma^2}\sigma_{\epsilon\lambda}\right)\omega^\lambda + (\rho + p)\omega_\epsilon + + \frac{1}{2}\eta_\epsilon{}^{\alpha\beta\gamma}V_\gamma\left[q_{\alpha;\beta} + \pi^\lambda{}_\alpha(\sigma_{\beta\lambda} - \omega_{\beta\lambda})\right], \tag{123}$$

With the Lifshitz method, we can decompose the gauge-invariant variable set in terms of the basis $Q(x^\alpha)$ and analyze the stability of the dynamical system for the scalar perturbation. This analysis will not be pursued in this work; however, the reader is being referred to Klippert [83] for more details.

3.3 Kasner Solution

Kasner universes constitute a paradigm of the Bianchi-type I anisotropic space-times. We shall follow the procedure in the previous section and present a minimal closed set of gauge-independent observables for an adequate basis built for this specific background. We shall subsequently employ that basis in a dynamical system written in the framework of the quasi-Maxwellian equations. We will then find out that the method can be carried out to its end and yield a closed dynamical system. Although all three types of perturbation—scalar, vectorial and tensorial—can be presented and discussed in the same way, we will here limit our analysis to scalar perturbations. An extensive discussion of anisotropic cosmological models can be found in Ellis and van Elst [51].

The seminal work of Belinsky, Khalatnikov and Lifshitz [13, 14] has shown that—for any kind of regular matter satisfying the usual energy conditions (cf. Hawking and Ellis [64]) in the neighborhood of a singularity—the Bianchi-type I Kasner solution works as an attractor for all the other solutions. In this sense, these geometries are good paradigms for anisotropic models, which have been extensively analyzed in the scientific literature (see Novello and Duque [112], Novello and de Freitas [113]).

The problem of the stability of anisotropic cosmological models and the analysis of perturbations has also been extensively studied in the literature using the method based on the perturbations of the metric tensor and dynamics determined by Einstein's equations by Matarrese [96, 97], Myoedema [98], Noh [107], Ibáñez [74] and Mutoh [105].

In order to apply linear perturbation theory, we will obtain a basis analogous to the spherical harmonics bases. We will then study the dynamical system in the framework of the quasi-Maxwellian (QM) equations. However, a slight change in the method will be necessary at this point, given the existence of such non-null tensorial quantities as the shear and the Weyl tensor in the background, analogous to the Bardeen [9] variables. The dynamical equations for these extra quantities are obtained from the QM equations.

3.3.1 The Anisotropic Basis

In order to make the temporal dependence of the perturbations explicit in the quasi-Maxwellian (QM) equations, we have to obtain a basis in terms of which all perturbed quantities can be expressed. Since we are dealing here with an anisotropic background, we shall avoid the spherical harmonics and will have to construct a new basis.

In this section, the three types of bases—scalar, vectorial and tensorial—will be exhibited. Since we are considering a background free of any matter, an apparent difficulty appears regarding the adequacy of analyzing matter-related perturbations. However, it is possible to write a generalized solution for the specific Kasner model, which presents matter-related terms that are of a lower order than the geometrical ones. It has been argued that the matter-related terms do not contribute to the unperturbed background, we nonetheless see that they might have an important contribution after perturbation.

The Scalar Basis To be able to obtain a scalar basis $\{Q(x, y, z)\}$ for the Kasner background, we will impose the equation

$$\hat{\nabla}^2 Q = n^2 Q, \tag{124}$$

where n^2 is a function of time.

In Cartesian coordinates, this equation is integrated to give

$$Q(x, y, z) = N \exp[-i(n_1 x + n_2 y + n_3 z)], \tag{125}$$

where N and n_j (for $j = 1, 2, 3$) are arbitrary constants and we infer the relation $n^2 = -h_{\alpha\beta} n^\alpha n^\beta$.

Using (124) and (125), we proceed to write the vector and the symmetric traceless second-order tensor bases, which will define the corresponding perturbed quantities

$$\begin{aligned} \hat{Q}_\alpha &\equiv \hat{\nabla}_\alpha Q = -i n_\alpha Q, \\ \hat{Q}_{\alpha\beta} &\equiv \hat{\nabla}_\beta Q_\alpha - \tfrac{1}{3} n^2 h_{\alpha\beta} Q, \end{aligned} \tag{126}$$

which directly shows the symmetry and the trace-free properties the tensor.

From the vector definition, $\hat{Q}_\alpha$ has only spatial components. The tensor $\hat{Q}_{\alpha\beta}$, written in terms of the scalar Q, has the form

$$\hat{Q}_{\alpha\beta} = -\left(n_\alpha n_\beta + \frac{n^2}{3} h_{\alpha\beta}\right) Q. \tag{127}$$

The following properties are then obtained:

$$\begin{aligned} \hat{\nabla}^\alpha \hat{Q}_\alpha &= n^2 Q, \\ \hat{\nabla}^\mu \hat{Q}_{\mu\nu} &= \frac{2}{3} n^2 \hat{Q}_\nu, \\ \hat{\nabla}^2 \hat{Q}_\alpha &= n^2 \hat{Q}_\alpha, \\ \hat{\nabla}^2 \hat{Q}_{\alpha\beta} &= n^2 \hat{Q}_{\alpha\beta} \\ \dot{Q} &= 0, \\ (\hat{Q}_\alpha)^{\cdot} &= \left(\sigma_\alpha{}^\beta + \frac{\theta}{3} h_{\alpha\beta}\right) \hat{Q}_\beta, \\ (\hat{Q}_{\alpha\beta})^{\cdot} &= -\frac{2}{3}\theta \hat{Q}_{\alpha\beta} - \sigma_{(\alpha}{}^\mu \hat{Q}_{\beta)m} - \frac{2}{3} n^2 \sigma_{\alpha\beta} Q \\ &\quad + \frac{2}{3} h_{\alpha\beta} \sigma^{\mu\nu} \hat{Q}_{\mu\nu}. \end{aligned} \tag{128}$$

We now choose a specific direction of propagation for the scalar basis.[4] We therefore set

$$n_1 = n_2 = 0 \Rightarrow n^2 = t^{-2p_3} (n_3)^2, \tag{129}$$

The scalar basis and its correlated quantities then take the very simple form:

$$\begin{aligned} Q &= N e^{-i n_3 z}, \\ \hat{Q}_\alpha &= -i n_3 (0, 0, Q), \\ \hat{Q}_{\alpha\beta} &= \tfrac{n^2}{3} Q \begin{pmatrix} t^{2p_1} & 0 & 0 \\ 0 & t^{2p_2} & 0 \\ 0 & 0 & -2t^{2p_3} \end{pmatrix}. \end{aligned} \tag{130}$$

The Vectorial Basis In analogy with the scalar case, for the vector basis $\left\{\hat{P}_\alpha\right\}$, we impose the equation

$$\hat{\nabla}^2 \hat{P}_\alpha = m^2 \hat{P}_\alpha, \tag{131}$$

where m^2 is a function of t.

Integration of this equation leads to the equality

$$\hat{P}_\alpha = \mathcal{P}^0{}_\alpha e^{-i m_j x^j}. \tag{132}$$

We choose $\mathcal{P}^0{}_\alpha$ and m_j as constants.[5] From (131) and (132), it follows that

$$m^2 = -h^{\alpha\beta} m_\alpha m_\beta. \tag{133}$$

[4]This procedure was adopted in several instances in the literature. See, for example, Sagnotti [145].

[5]This choice is necessary to avoid that complex terms or explicit dependences on spatial coordinates that occur when the simple derivatives of the vector basis are calculated.

For $\left\{\hat{P}_\alpha\right\}$ to a basis, two properties must be valid. The first one concerns the fact that the $\hat{P}_\alpha$ must be spatial quantities. This is immediately satisfied with the choice $\mathcal{P}^0{}_0 = 0$, or

$$V^\alpha \hat{P}_\alpha = 0. \tag{134}$$

This property must also be preserved in time, i.e.,

$$(V^\alpha \hat{P}_\alpha)^{\cdot} = 0, \tag{135}$$

which is identically valid for $\mathcal{P}_\mu{}^0$ =const.

The second property, namely that no scalar quantities can be obtained from the vector $\hat{P}_\alpha$, implies that

$$\hat{\nabla}^\alpha \hat{P}_\alpha = 0, \tag{136}$$

and can be written as

$$h^{\alpha\beta} m_\alpha \hat{P}_\beta = 0. \tag{137}$$

This property must also be conserved in time; hence it follows that

$$\hat{\nabla}^\alpha \hat{P}_\alpha = 0 \Longrightarrow \sigma^{\alpha\beta} m_\alpha \hat{P}_\beta = 0. \tag{138}$$

The conditions (137) and (138) can also be written in terms of the $\mathcal{P}^0{}_\alpha$:

$$t^{-2p_1} m_1 \mathcal{P}^0_x + t^{-2p_2} m_2 \mathcal{P}^0_y + t^{-2p_3} m_3 \mathcal{P}^0_z = 0 \tag{139}$$

and

$$p_1 t^{-2p_1} m_1 \mathcal{P}^0_x + p_2 t^{-2p_2} m_2 \mathcal{P}^0_y + p_3 t^{-2p_3} m_3 \mathcal{P}^0_z = 0. \tag{140}$$

From $\mathcal{P}_\alpha$, it is possible to construct three quantities: a symmetric, traceless second-order tensor (which we will denote $\hat{P}_{\alpha\beta}$), a pseudo-vector denoted $\hat{P}^*_\alpha$ and finally the corresponding pseudo-tensor $\hat{P}^*_{\alpha\beta}$. The definitions for these quantities are the following:

$$\begin{aligned} \hat{P}_{\alpha\beta} &\equiv \hat{\nabla}_{(\alpha} \hat{P}_{\beta)}, \\ \hat{P}^*_\alpha &\equiv \eta_\alpha{}^{\beta\mu\nu} V_\beta (\hat{\nabla}_\nu \hat{P}_\mu), \\ \hat{P}^*_{\alpha\beta} &\equiv \hat{\nabla}_{(\alpha} \hat{P}^*{}_{\beta)}, \end{aligned} \tag{141}$$

respectively.

The first of (141) is immediately rewritten as

$$\hat{P}_{\alpha\beta} = -i m_{(\alpha} \hat{P}_{\beta)} \tag{142}$$

and its corresponding Laplacian and time-projected derivative are proven to be given by

$$\hat{\nabla}^2 \hat{P}_{\alpha\beta} = m^2 \hat{P}_{\alpha\beta}, \tag{143}$$

$$\left(\hat{P}_{\alpha\beta}\right)^{\cdot} = -\frac{2}{3}\theta \hat{P}_{\alpha\beta} - \sigma^\gamma{}_{(\alpha} \hat{P}_{\beta)\gamma}. \tag{144}$$

The pseudo-vector $\hat{P}^*_\alpha$, from the second definition in (141), is

$$\hat{P}^*_\alpha = -i \eta_\alpha{}^{\beta\mu\nu} V_\beta m_\nu \hat{P}_\mu \tag{145}$$

Since all these quantities describe the perturbations, it follows that the same properties that define $\hat{P}_\alpha$ should also be valid for $\hat{P}^*_\alpha$. Therefore, the pseudo-vector should be both a spatial and a divergence-free quantity, which it is. These conditions must be preserved in time, which is identically valid for the first property. The condition of preservation for the null divergence property is given by the equality

$$\left(\hat{\nabla}^\alpha \hat{P}^*_\alpha\right)^{\cdot} = 0 \Longrightarrow \sigma^{\alpha\beta} m_\alpha \hat{P}^*_\beta = 0, \tag{146}$$

which is then rewritten in terms of the $\mathcal{P}^0_\alpha$ as

$$\begin{aligned} (p_2 - p_3) m_2 m_3 \mathcal{P}^0_x + (p_3 - p_1) m_1 m_3 \mathcal{P}^0_y \\ + (p_1 - p_2) m_1 m_2 \mathcal{P}^0_z = 0 \end{aligned} \tag{147}$$

In addition to the above condition, the following useful results are obtained:

$$\hat{\nabla}^2 \hat{P}^*_\alpha = m^2 \hat{P}^*_\alpha, \tag{148}$$

and

$$\left(\hat{P}^*_\alpha\right)^{\cdot} = -\frac{1}{3}\theta \hat{P}^*_\alpha - \sigma^\beta{}_\alpha \hat{P}^*_\beta. \tag{149}$$

The last quantity to be considered is the symmetric, traceless pseudo-tensor $\hat{P}^*_{\alpha\beta}$. From the third definition (141), we get

$$\hat{P}^*_{\alpha\beta} = -i m_{(\alpha} \hat{P}^*_{\beta)}, \tag{150}$$

and the relations below follow immediately:

$$\hat{\nabla}^\alpha \hat{P}^*_{\alpha\beta} = m^2 \hat{P}^*_{\alpha\beta}, \tag{151}$$

$$\hat{\nabla}^2 \hat{P}^*_{\alpha\beta} = m^2 \hat{P}^*_{\alpha\beta}, \tag{152}$$

and

$$\left(\hat{P}^*_{\alpha\beta}\right)^{\cdot} = -\frac{2}{3}\theta \hat{P}^*_{\alpha\beta} - \sigma^\gamma{}_{(\alpha} \hat{P}^*_{\beta)\gamma}. \tag{153}$$

The most general form for the vectorial basis implies obtaining suitable $\mathcal{P}^0_\alpha$ and m_j and replacing them in the basis expression (132). These quantities are defined from the conditions (139), (140) and (147) as[6]

$$m_1 \equiv m_2 = 0, \quad \mathcal{P}_z^0 = 0, \tag{154}$$

and, using (154) in (132), we come to the result

$$\hat{P}_\alpha = e^{-im_3 z}\left(\mathcal{P}_x^0, \mathcal{P}_y^0, 0\right), \tag{155}$$

where m_3, $\mathcal{P}_x^0$ and $\mathcal{P}_y^0$ are arbitrary constants.

In addition, the quantities $\hat{P}_{\alpha\beta}$, $\hat{P}_\alpha^*$ and $\hat{P}_{\alpha\beta}^*$ are written, in this case, as

$$\hat{P}_{\alpha\beta} = -im_3 e^{-im_3 z}\begin{pmatrix} 0 & 0 & \mathcal{P}_x^0 \\ 0 & 0 & \mathcal{P}_y^0 \\ \mathcal{P}_x^0 & \mathcal{P}_y^0 & 0 \end{pmatrix}, \tag{156}$$

$$\hat{P}_\alpha^* = i\eta^{0123} m_3 e^{-im_3 z}\left(-t^{2p_1}\mathcal{P}_y^0, t^{2p_2}\mathcal{P}_x^0, 0\right), \tag{157}$$

and

$$\hat{P}_{\alpha\beta}^* = \eta^{0123}(m_3)^2 e^{-im_3 z}\begin{pmatrix} 0 & 0 & -t^{2p_1}\mathcal{P}_y^0 \\ 0 & 0 & t^{2p_2}\mathcal{P}_x^0 \\ -t^{2p_1}\mathcal{P}_y^0 & t^{2p_2}\mathcal{P}_x^0 & 0 \end{pmatrix}. \tag{158}$$

The Tensorial Basis We will begin by defining the tensor $\hat{U}^\mu{}_\nu(t, x, y, z)$, which is written in matrix form as

$$\hat{U}^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \psi & \phi \\ 0 & \eta & \beta & \epsilon \\ 0 & \chi & \zeta & \gamma \end{pmatrix}, \tag{159}$$

where $(\alpha, \psi, \phi, \eta, \beta, \epsilon, \chi, \zeta, \gamma)$ are functions of all four coordinates.

Our choice for the tensorial basis enables us to simplify future calculations somewhat. It is easy to write the totally contravariant tensor in the form

$$\hat{U}^\mu{}_\nu = h^{\mu\alpha}\hat{U}_{\alpha\nu} \tag{160}$$

with $\hat{U}_{\alpha\nu} = \hat{U}_{\nu\alpha}$.

To be a basis, $\hat{U}^\mu{}_\nu$ has to be a solution of the equation

$$\hat{\nabla}^2\hat{U}^\mu{}_\nu = k^2\hat{U}^\mu{}_\nu, \tag{161}$$

where k^2 is a function of time.

Solving this equation, we obtain an explicit form for the tensorial basis

$$\hat{U}^\mu{}_\nu = \mathcal{U}^\mu{}_\nu e^{-ik_j x^j}, \tag{162}$$

where the $\mathcal{U}^\mu{}_\nu$ will be taken as covariantly constant tensors and k_j are arbitrary constants related to the wave number k^2 and the components of the metric tensor through the following relation:[7]

$$k^2 = -h^{jl}k_j k_l = \left[\left(\frac{k_1}{a(t)}\right)^2 + \left(\frac{k_2}{b(t)}\right)^2 + \left(\frac{k_3}{c(t)}\right)^2\right]. \tag{163}$$

As in the previous cases, the tensor basis must obey the following properties:

(*I*) The tensor basis should be orthogonal to V_α:

$$V_\mu\hat{U}^\mu{}_\nu = 0. \tag{164}$$

(*II*) Scalars cannot be obtained from the tensor basis:

$$h^{\mu\nu}\hat{U}_{\mu\nu} = \hat{U}^\mu{}_\mu = 0, \tag{165}$$

or, using (159),

$$\alpha + \beta + \gamma = 0. \tag{166}$$

(*III*) Vectors cannot be obtained from the tensor basis:

$$\hat{\nabla}^\mu\hat{U}_{\mu\nu} = 0, \tag{167}$$

which gives

$$k_\mu\hat{U}^\mu{}_\nu = 0, \tag{168}$$

Equation (168) above can be rewritten, using (162), as

$$k_1\alpha + k_2\eta + k_3\chi = 0, \quad k_1\psi + k_2\beta + k_3\zeta = 0, \quad k_1\phi + k_2\epsilon + k_3\gamma = 0, \tag{169}$$

It is easily seen that all the properties above are preserved in time.

At this point, it becomes necessary to define a quantity that enables us to write pseudo-tensorial perturbations. Therefore, we define the dual $\hat{U}^{*\mu}{}_\nu$ as

$$\hat{U}^*_{\mu\nu} \equiv \frac{1}{2}h^\alpha_{(\mu}h^\beta_{\nu)}\eta_\beta{}^{\lambda\epsilon\gamma}V_\lambda(\hat{\nabla}_\epsilon\hat{U}_{\gamma\alpha}), \tag{170}$$

which can be then rewritten as

$$\hat{U}^{*\mu}{}_\nu \equiv -\frac{i}{2}k_\epsilon\left(\eta_\gamma{}^{\lambda\epsilon\mu}V_\lambda\hat{U}^\gamma{}_\mu + \eta^{\gamma\lambda\epsilon}{}_\nu V_\lambda\hat{U}^\mu{}_\gamma\right). \tag{171}$$

It follows that all the properties of the tensorial basis $\hat{U}^\mu{}_\nu$ are equally valid for the dual tensor $\hat{U}^{*\mu}{}_\nu$ and that they are preserved in time.

[6]Equation (154) calls for a choice of a specific direction for the basis, which has been made a number of times in the literature. See, for example, [145].

[7]This choice was made to avoid spatially-dependent terms when calculating the derivative $(\hat{U}^\mu{}_\nu)^\cdot$

We can obtain an explicit form for $\hat{U}^{\mu}{}_{\nu}$. From (60) and (159), we have that

$$\hat{U}^2{}_1 \equiv \eta = g_{11} g^{22} \hat{U}^1_2 \equiv t^{2(p_1-p_2)}\psi,$$
$$\hat{U}^3{}_1 \equiv \chi = g_{11} g^{33} \hat{U}^1_3 \equiv t^{2(p_1-p_3)}\phi, \tag{172}$$
$$\hat{U}^3{}_2 \equiv \zeta = g_{22} g^{33} \hat{U}^2_3 \equiv t^{2(p_2-p_3)}\epsilon.$$

Using the above results and the null trace condition (166) in condition (169), we find that

$$\begin{aligned} &k_1\alpha + k_2 t^{2(p_1-p_2)}\psi + k_3 t^{2(p_1-p_3)}\phi = 0,\\ &k_1\psi + k_2\beta + k_3 t^{2(p_2-p_3)}\epsilon = 0,\\ &k_1\phi + k_2\epsilon - k_3(\alpha+\beta) = 0. \end{aligned} \tag{173}$$

However, since both the k_j and the $\mathcal{U}^{\mu}{}_{\nu}$ are constant, we can see that each term in the three above relations must also be constants. We then choose a specific direction for the basis [145], taking

$$\begin{aligned} &k_1 = k_2 = 0,\\ &k_3 \neq 0. \end{aligned} \tag{174}$$

Then, (173) simplify to

$$\begin{aligned} &t^{2(p_1-p_3)}\phi = 0,\\ &t^{2(p_2-p_3)}\epsilon = 0,\\ &\alpha+\beta = 0, \end{aligned} \tag{175}$$

and give, as a consequence, the following equalities:

$$\begin{aligned} &\phi = \epsilon = 0,\\ &\beta = -\alpha \Rightarrow \gamma = 0. \end{aligned} \tag{176}$$

Therefore, using (164), the spatial components of the tensorial basis and its dual are written in matrix form as

$$\hat{U}^{\mu}{}_{\nu} = \mathcal{U}^{\mu}{}_{\nu} e^{-ik_3 z} = \begin{pmatrix} \alpha & \psi & 0 \\ \eta & -\alpha & 0 \\ \chi & \zeta & 0 \end{pmatrix}, \tag{177}$$

$$\hat{U}^{*\mu}{}_{\nu} = -\frac{i}{2}\eta^{0123} k_3 \begin{pmatrix} \left(t^{2p_1}\psi + t^{2p_2}\eta\right) & -2t^{2p_2}\alpha & 0 \\ -2t^{2p_1}\alpha & -\left(t^{2p_1}\psi + t^{2p_2}\eta\right) & 0 \\ t^{2p_1}\chi & -t^{2p_2}\zeta & 0 \end{pmatrix}. \tag{178}$$

3.3.2 The Gauge-Invariant Variables and Their Dynamics

The dynamical system for an anisotropic background will be obtained in the framework of the quasi-Maxwellian (QM) equations. In the Kasner background, the QM equations are reduced to the set

$$(\sigma_{ij})^{\dot{}} + E_{ij} + \frac{2}{3}\theta\sigma_{ij} - \frac{1}{3}(2\sigma^2)h_{ij} + \sigma_{ik}\sigma^k{}_j = 0, \tag{179}$$

$$(E_{ij})^{\dot{}} + 3\theta\ E_{ij} + \frac{3}{2}\sigma^{\mu}{}_i E_{j\mu} - h_{ij}\sigma^{\mu\nu}E_{\mu\nu} = 0, \tag{180}$$

and

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 = 0, \tag{181}$$

The natural step is, then, to write the perturbed QM equations, by making the usual choice: $A_{(pert.)} = A_{(back.)} + (\delta A)$. However, a modification in this method becomes necessary at this point: the three non-null quantities in the Kasner background ($\sigma_{\mu\nu}$, $E_{\mu\nu}$ and θ) should be replaced by "artificial" quantities in order to eliminate all gauge-dependent terms from the dynamical system equations. These new variables are written in terms of the original, gauge-dependent variables. Nevertheless, they constitute "good" quantities, in the sense that they are zero in the Kasner background, as per the Stewart lemma [148]. This procedure is analogous to the one implemented by Bardeen [9], but the variables obtained in the present case are much simpler, as we shall see in the next section.

Minimal Closed Set of Variables for the Kasner Background The starting point to obtain the new variables is the set of QM equations for the Kasner background, (179)–(181). If we employ the following relations, which are easily demonstrated and specifically valid for the Kasner background:

$$\begin{aligned} (\sigma^{\alpha}{}_{\beta})^{\dot{}} &= -\theta\,\sigma^{\alpha}{}_{\beta},\\ (E^{\alpha}{}_{\beta})^{\dot{}} &= -2\,\theta\ E^{\alpha}{}_{\beta},\\ \dot{\theta} &= -\theta^2, \end{aligned} \tag{182}$$

we will be able to define the new variables that are to replace the original ones as

$$X^{\alpha}{}_{\beta} \equiv E^{\alpha}{}_{\beta} - \frac{1}{3}\theta\sigma^{\alpha}{}_{\beta} - \frac{2\sigma^2}{3}h^{\alpha}{}_{\beta} + \sigma^{\alpha}{}_{\mu}\sigma^{\mu}{}_{\beta}, \tag{183}$$

$$Y^{\alpha}{}_{\beta} \equiv \theta E^{\alpha}{}_{\beta} + \frac{3}{2}\sigma^{\alpha}{}_{\mu}E^{\mu}{}_{\beta} + \frac{3}{2}\sigma^{\mu}{}_{\beta}E^{\alpha}{}_{\mu} - h^{\alpha}{}_{\beta}\sigma^{\mu\nu}E_{\mu\nu}, \tag{184}$$

and

$$W \equiv 2\sigma^2 - \frac{2}{3}\theta^2. \tag{185}$$

These three variables are easily proven to be zero for the Kasner background and, therefore, "good" ones to be perturbed. They may, then, replace the shear, electric part of the Weyl tensor and expansion as the new variables in the dynamic system. An additional simplifying choice will be made: the relation between the energy density ρ and the pressure p is

$$p = \lambda\rho, \quad \lambda \equiv \text{const.}, \tag{186}$$

even after the background is perturbed.

This choice, which has also been made for the FLRW case (see Novello [122–124]), will be considered as valid throughout this analysis. The complete minimal closed set of variables to appear in the dynamical system is therefore[8]

$$\mathcal{M} = \{X_{\alpha\beta}, Y_{\alpha\beta}, H_{\alpha\beta}, \pi_{\alpha\beta}, q_\alpha, a_\alpha, \omega_\alpha, W, \rho\}. \quad (187)$$

The next step in the analysis is to obtain the complete dynamics for the new variables, $X_{\alpha\beta}$, $Y_{\alpha\beta}$, W. The resulting set of equations and the remaining QM equations must, then, be rewritten in terms of the new gauge-independent variables in M. This constitutes the dynamical system of equations used to study the perturbations of the Kasner model. Such a complete dynamical system as well as the steps for its derivation can be seen in Novello et al. [125]. The next sections will deal with the three perturbation cases and the results obtained for each.

3.3.3 Scalar Perturbations

In this case, the minimal closed set of observables $\mathcal{M}$ involves practically all the original variables of the system, which we proceed to present here in terms of the scalar basis Q:

$$\begin{array}{ll} (\delta X_{\alpha\beta}) = X(t)\hat{Q}_{\alpha\beta}, & (\delta Y_{\alpha\beta}) = Y(t)\hat{Q}_{\alpha\beta}, \\ (\delta\pi_{\alpha\beta}) = \pi(t)\hat{Q}_{\alpha\beta}, & (\delta q_\alpha) = q(t)\hat{Q}_\alpha, \\ (\delta a_\alpha) = \psi(t)\hat{Q}_\alpha, & (\delta\rho) = R(t)Q, \\ (\delta W) = W(t)Q, & (\delta p) = p(t)Q, \end{array} \quad (188)$$

where the spatial part of the velocity, (δV_k), is also zero in the background (but not an adequate variable, since its value in the background depends on the choice of an observer) and it is written as

$$(\delta V_k) = V(t)\hat{Q}_k. \quad (189)$$

Since we are dealing with scalar perturbations, the vorticity and related perturbations, for instance the magnetic part of the Weyl tensor, are not defined—details in Novello et al. [122] and Goode [59]. The relation between the shear and the anisotropic pressure is still valid, but in this case, the viscosity ξ is also a "good" variable (i.e., it is a gauge-independent variable for it is zero in the Kasner background), written as

$$(\delta\xi) = \xi(t)\ Q,$$

which must be considered.

From (188) and (127), it is possible to obtain $(\delta\xi)$ in terms of $(\delta\pi_{\alpha\beta})$ as

$$\xi(t) = -\frac{1}{(2\sigma^2)}(\sigma^{\mu\nu}n_\mu n_\nu)\pi(t). \quad (190)$$

[8]Since it has no dynamical equation of its own, the acceleration, i.e., the variable α, must be eliminated to make the dynamical system closed. This will be achieved by fixing a value for the function (δa_α).

As in the previous cases, we take the perturbed anisotropic pressure $(\delta\pi_{\alpha\beta})$ as zero, in order to simplify the dynamical system to be solved. Thus,

$$\pi(t) = \xi(t) = 0. \quad (191)$$

Using the results for the scalar basis, we then obtain the dynamical system for scalar perturbations

$$\left(\dot{X} + \theta X - Y + \frac{1}{2}q + \frac{1}{3}\theta\psi\right)\hat{Q}_{\mu\nu} + \left(-\frac{2}{3}n^2 X + \frac{1}{3}n^2\psi + \frac{1}{3}(W+R)\right)[\sigma_{\mu\nu}Q] + - \psi\left[\sigma^\alpha{}_{(\mu}\hat{-}Q_{\nu)\alpha} - \frac{2}{3}h_{\mu\nu}\sigma^{\alpha\beta}\hat{Q}_{\alpha\beta}\right] = 0, \quad (192)$$

$$\left(\dot{Y} + \frac{4}{3}\theta Y + \frac{1}{2}\theta q\right)\hat{Q}_{\mu\nu} + \left[-\frac{2}{3}n^2 Y + \theta(1+\lambda)R\right][\sigma_{\mu\nu}Q] + (n^2 q - 2n^2\psi + W - R)[E_{\mu\nu}Q] + \frac{1}{2}\left(-5Y + \frac{3}{2}q\right)\left(\sigma^\alpha{}_{(\mu}\hat{Q}_{\nu)\alpha} - \frac{2}{3}h_{\mu\nu}\sigma^{\alpha\beta}\hat{Q}_{\alpha\beta}\right) + \frac{3}{2}(X-\psi)\left[E^\alpha{}_{(\mu}\hat{Q}_{\nu)\alpha} - \frac{2}{3}h_{\mu\nu}E^{\alpha\beta}\hat{Q}_{\alpha\beta}\right] = 0, \quad (193)$$

$$\dot{W} + \frac{2}{3}\theta W - 2(\sigma^{\alpha\beta}n_\alpha n_\beta)X + 2\left[(\sigma^{\alpha\beta}n_\alpha n_\beta) + \frac{2}{3}\theta n^2\right]\psi - \frac{2}{3}\theta(1+3\lambda)R = 0, \quad (194)$$

$$\left\{[p_1(1-p_1) - p_2(1-p_2)] + \frac{1}{12}(1-3p_3)(p_1-p_2)\right\}\psi = 0, \quad (195)$$

$$\dot{R} + \theta(1+\lambda)R + n^2 q = 0, \quad (196)$$

$$\dot{q} + \theta q - \lambda t^{-8/3}R = 0, \quad (197)$$

and

$$\left[\frac{2}{3}n^2 X - \theta q + \frac{1}{3}(W-R)\right]\hat{Q}_\alpha + \frac{1}{2}\psi\left[\left(\sigma^\alpha{}_\gamma\sigma^{\gamma\beta} + \frac{4}{3}\theta\sigma_\alpha{}^\beta\right)\hat{Q}_\beta\right] = 0. \quad (198)$$

Equation (195) is satisfied in two cases: (1) $p_1 \neq p_2 \Rightarrow \psi = 0$ and (2) $p_1 = p_2 \Rightarrow$ isotropy plane and the simplest choice here is (1), with $\psi = 0$.

The system therefore becomes

$$\left(\dot{X}+\theta X-Y+\frac{1}{2}q\right)\hat{Q}_{\mu\nu}+\left(-\frac{2}{3}n^2X+\frac{1}{3}(W+R)\right)[\sigma_{\mu\nu}Q]=0, \tag{199}$$

$$\left(\dot{Y}+\frac{4}{3}\theta Y+\frac{1}{2}\theta q\right)\hat{Q}_{\mu\nu}+\left[-\frac{2}{3}n^2Y+\theta(1+\lambda)R\right][\sigma_{\mu\nu}Q] + (n^2q+W-R)[E_{\mu\nu}Q] + \frac{1}{2}\left(-5Y+\frac{3}{2}q\right)\left(\sigma^{\alpha}{}_{(\mu}\hat{Q}_{\nu)\alpha}-\frac{2}{3}h_{\mu\nu}\sigma^{\alpha\beta}\hat{Q}_{\alpha\beta}\right) + \frac{3}{2}X\left[E^{\alpha}{}_{(\mu}\hat{Q}_{\nu)\alpha}-\frac{2}{3}h_{\mu\nu}E^{\alpha\beta}\hat{Q}_{\alpha\beta}\right]=0, \tag{200}$$

$$\dot{W}+\frac{2}{3}\theta W-2(\sigma^{\alpha\beta}n_\alpha n_\beta)X-\frac{2}{3}\theta(1+3\lambda)R=0, \tag{201}$$

$$\dot{R}+\theta(1+\lambda)R+n^2q=0, \tag{202}$$

$$\dot{q}+\theta q-\lambda t^{-8/3}R=0, \tag{203}$$

and

$$\left[\frac{2}{3}n^2X-\theta q+\frac{1}{3}(W-R)\right]\hat{Q}_\alpha=0. \tag{204}$$

Although (199) and (200) cannot be factored out in the scalar basis, the remaining equations in the system, (201) and (204), can be separately integrated. The constraint (204) eliminates the variable X from the reduced system

$$X=\frac{3}{2}\frac{\theta}{n^2}t^{8/3}q+\frac{1}{2n^2}R-\frac{1}{2n^2}W, \tag{205}$$

so that the dynamics of W, (201), is written as

$$\dot{W}+\left[\frac{2}{3}\theta W+\frac{1}{n^2}(\sigma^{\alpha\beta}n_\alpha n_\beta)\right]W-\left[\frac{2}{3}\theta(1+3\lambda)+(\sigma^{\alpha\beta}n_\alpha n_\beta)\right]\times R-3\frac{\theta}{n^2}t^{8/3}(\sigma^{\alpha\beta}n_\alpha n_\beta)q=0, \tag{206}$$

and, with the relation

$$\frac{1}{n^2}(\sigma^{\alpha\beta}n_\alpha n_\beta)=\frac{1}{3}\theta(1-p_3), \tag{207}$$

Table 2 Stability analysis results for scalar perturbations

Value for x	$q(t)$	$R(t)$ and $W(t)$
$x<-(8/3+\lambda)$	Stable	Stable
$-1<x<0$	Stable	Unstable (faster than $t^{2/3}$)
$x=0$	Constant	Unstable (faster than $t^{5/3}$)
$x>0$	Unstable	Unstable (faster than $t^{5/3}$)

the final dynamics for W takes the form

$$\dot{W}+(1-p_3)\theta W-(1+2\lambda-p_3)\theta R+(3p_3-1)\theta t^{8/3}q=0, \tag{208}$$

and the reduced dynamical system, which is closed in the variables W, R and q, is given by (202), (203) and (208). As before, this reduced system can be solved for the following Ansatz:

$$q(t)=q_0t^x,\quad R(t)=R_0t^y,\quad W(t)=W_0t^w, \tag{209}$$

where q_0, R_0, W_0 and x, y, w are constants to be determined.

Inspection of the powers of t in the three equations immediately yields the relation between x, y and w

$$y=w=x+\frac{5}{3}, \tag{210}$$

while the rest of the equations gives the following results:

$$R_0=\frac{1}{\lambda}(x+1)q_0, \tag{211}$$

and

$$W_0=[(x+1)(1+2\lambda-p_3)+\lambda(1-3p_3)](w+1-p_3)^{-1}q_0, \tag{212}$$

where $w\neq(p_3-1)$ and

$$(n_3)^2=\frac{1}{\lambda}(x+1)\left(x+\frac{8}{3}+\lambda\right). \tag{213}$$

The above results are inapplicable to case $\lambda=0$; the details of this specific solution can be seen in [125].

In order to analyze the stability of the above solution, we will impose that $(n_3)^2$ be positive. From (213), letting $\lambda > 0$,[9] we are led to two possibilities: (1) $x > -1$; (2) $x < -(8/3 - \lambda)$. Table 2 list the results for each case.

We see that, although unstable solutions exist, as we will see in the case of tensorial perturbations, the perturbations are not catastrophic (a power law divergence). We also see that the matter density is unstable for any choice of the constant exponent x that implies $(n_3)^2$ positive.

This concludes the analysis for scalar perturbations ($\lambda \neq 0$). The case $\lambda = 0$ must now be studied. For $\lambda = 0$, the reduced dynamical system is given by the equalities

$$\dot{W}+(1-p_3)\theta W-(1-p_3)\theta R+(3p_3-1)\theta^2 t^{8/3} q = 0, \quad (214)$$

$$\dot{R} + \theta R + n^2 q = 0, \quad (215)$$

and

$$\dot{q} + \theta q = 0. \quad (216)$$

Equation (216) can be directly integrated as

$$q(t) = q_0 t^{-1}, \quad (217)$$

where q_0 is a constant, while the Ansatz

$$R(t) = R_0 t^y, \quad W(t) = W_0 t^w, \quad (218)$$

can be used in (214) and (215) to give the expression

$$y = w = \frac{2}{3}, \quad W_0 = \frac{(1-p_3)}{(w+1-p_3)} R_0, \quad q_0 = \frac{5}{3(n_3)^2} R_0.$$

The above results yield a partially unstable solution for $\lambda = 0$, with R and W divergent in t, while $q \to 0$ for $t \to \infty$.

3.3.4 Vectorial Perturbations

The perturbations associated with the state of motion of a fluid (energy-density perturbations not taken into account) are in principle described by the following minimal closed set of variables:[10]

$$\mathcal{M} = \left\{ X_{\alpha\beta}, Y_{\alpha\beta}, H_{\alpha\beta}, q_\alpha, a_\alpha, \omega_\alpha \right\}. \quad (219)$$

The perturbed quantities can then be written in terms of the vectorial basis $\hat{P}_\alpha$ as follows:

$$\begin{array}{ll} (\delta X_{\alpha\beta}) = X(t)\hat{P}_{\alpha\beta}, & (\delta Y_{\alpha\beta}) = Y(t)\hat{P}_{\alpha\beta}, \\ (\delta H_{\alpha\beta}) = H(t)\hat{P}^*_{\alpha\beta}, & (\delta q_\alpha) = q(t)\hat{P}_\alpha, \\ (\delta a_\alpha) = \psi(t)\hat{P}_\alpha, & (\delta\omega_\alpha) = \Omega(t)\hat{P}^*_\alpha, \\ (\delta V_k) = V(t)\hat{P}_k & \end{array}, \quad (220)$$

where the variable $V(t)$ is again inadequate, since it depends on the initial choice by an observer.

To successfully factor out the basis from the dynamical system equations, we have to choose between two possibilities: (1) eliminate one of the basis components, say $\mathcal{P}^0_y$ and (2) analyze solely the case of a background with an isotropy plane (namely, the Kasner solution).

Here, we choose (1) because it yields a more general result (and also contains the specific case (2), as we will presently show). The dynamical system for the vectorial case can then be obtained by factoring out the basis, as indicated by the following equations:

$$\dot{X} + \theta X - Y + \frac{m^2}{2} H + \frac{1}{4} q - \frac{1}{2}\theta p_2 \psi = 0, \quad (221)$$

$$\dot{Y} - \frac{1}{2}\theta(5p_2+1)Y + \frac{3}{2}\theta^2 p_2(p_2-1)X + \frac{3}{4}\theta m^2 (p_2-1)H - \frac{3}{8}\theta(p_2-1)q - \frac{3}{4}\theta^2 p_2(p_2-1)\psi = 0, \quad (222)$$

$$\dot{H} - \frac{1}{2}\theta(5p_1+1/3)H - \frac{1}{2}X + \frac{1}{4}\theta(3(p_3-p_2)+2/3)\Omega + \frac{1}{4}\frac{\theta^2}{m^2}(p_2-p_3)(3p_1+1/3)\psi = 0, \quad (223)$$

$$\dot{q} + \theta q = 0, \quad (224)$$

$$\dot{\Omega} + \frac{1}{3}\theta(6p_2+1)\Omega - \frac{1}{2}t^{-8/3}\psi = 0, \quad (225)$$

$$m^2 t^{-8/3} X + \theta m^2(p_2-1/3)H + \frac{1}{2}\theta m^2(p_1+1)\Omega - \theta q + \frac{1}{2}\theta^2(p_1-1/3)(p_1+1)\psi = 0, \quad (226)$$

[9] The case $\lambda < 0$ yields $(n_3)^2 < 0$ for all x and will therefore not be considered here.

[10] The same observation regarding the variable a_α that was stated for the tensorial case holds here as well.

and

$$m^2H-\theta(p_1-p_3)X+3\theta^2t^{8/3}p_2(p_2-1)\Omega+\frac{1}{2}q=0. \quad (227)$$

The system (221)–(227) is not closed because there is no dynamics for the acceleration $\psi(t)$. However, the constraint (226) can be employed to express this variable in terms of the other variables in $\mathcal{M}$ and close the remaining dynamical system for a given background, specified by the triad (p_1, p_2, p_3). To obtain a specific solution for the dynamical system, we must choose (p_1, p_2, p_3). Here, we discuss the specific case of the Kasner solution with an isotropy plane, which is algebraically simpler to solve. This choice, applied to (221)–(227), leads to the following dynamical system:

$$\dot{X}+\theta X-Y+\frac{m^2}{2}H+\frac{1}{4}q+\frac{1}{3}\theta\psi=0, \quad (228)$$

$$\dot{Y}+\frac{13}{6}\theta Y-\frac{1}{3}\theta^2X+\frac{1}{4}\theta m^2H+\frac{1}{8}\theta q+\frac{1}{6}\theta^2\psi=0, \quad (229)$$

$$\dot{H}+\frac{11}{6}\theta H-\frac{1}{2}X-\frac{1}{12}\theta\Omega+\frac{7}{12}\frac{\theta^2}{m^2}\psi=0, \quad (230)$$

$$\dot{q}+\theta q=0, \quad (231)$$

$$\dot{\Omega}+\frac{5}{3}\theta\Omega-\frac{1}{2}t^{-8/3}\psi=0, \quad (232)$$

$$m^2t^{-8/3}X+\frac{1}{3}\theta m^2H+\frac{5}{6}\theta m^2\Omega-\theta q+\frac{5}{18}\theta^2\psi=0, \quad (233)$$

and

$$m^2H-\theta X-\frac{2}{3}\theta^2t^{8/3}\Omega+\frac{1}{2}q=0. \quad (234)$$

The acceleration ψ is related to other variables by (233), which then closes the system

$$\psi=\frac{18}{5\theta}q-\frac{18}{5}\frac{m^2}{\theta^2}t^{-8/3}X-\frac{6}{5}\frac{m^2}{\theta}H-3\frac{m^2}{\theta}\Omega, \quad (235)$$

and the closed dynamical system is then written in the form

$$\dot{X}+\frac{1}{5\theta}(5\theta^2-6t^{-8/3}m^2)X-Y+\frac{1}{10}m^2H-m^2\Omega+\frac{29}{20}q=0, \quad (236)$$

$$\dot{Y}+\frac{13}{6}\theta Y-\frac{1}{15}\left(5\theta^2+9t^{-8/3}m^2\right)X+\frac{1}{20}\theta m^2H -\frac{1}{2}\theta m^2\Omega+\frac{29}{40}\theta q=0, \quad (237)$$

$$\dot{H}+\frac{17}{15}\theta H-\frac{26}{10}t^{-8/3}X-\frac{11}{6}\theta\Omega+\frac{21}{10}\frac{\theta}{m^2}q=0, \quad (238)$$

$$\dot{\Omega}+\frac{1}{6\theta}\left(10\theta^2-9t^{-8/3}m^2\right)\Omega-\frac{9}{5}\frac{m^2}{\theta^2}t^{-16/3}X -\frac{3}{5}\frac{m^2}{\theta}t^{-8/3}H-\frac{9}{5\theta}t^{-8/3}q=0, \quad (239)$$

$$\dot{q}+\theta q=0, \quad (240)$$

and

$$\theta X-m^2H+\frac{2}{3}\theta^2t^{8/3}\Omega-\frac{1}{2}q=0. \quad (241)$$

Equation (240) can be directly integrated in $q(t)$, giving

$$q(t)=q_0t^{-1}, \quad (242)$$

where q_0 is an integration constant.

The rest of the system can be solved for the following Ansatz:

$$\begin{aligned} X(t) &= X_0t^x,\\ Y(t) &= Y_0t^y,\\ H(t) &= H_0t^z,\\ \Omega(t) &= \Omega_0t^w, \end{aligned} \quad (243)$$

where $X_0, Y_0, H_0, \Omega_0, x, y, z,$ and w are constants to be determined from (236)–(241), along with the constant m_3. The exponents are easily obtained as

$$\begin{aligned} x &= 0,\\ y &= -1,\\ z &= w = -5/3, \end{aligned} \quad (244)$$

and the remaining constants must satisfy the following conditions:

$$\begin{aligned} &8[5-3(m_3)^2]X_0-20Y_0+2(m_3)^2H_0\\ &\qquad -20(m_3)^2\Omega_0+29q_0=0,\\ &8[5+9(m_3)^2]X_0-380Y_0-6(m_3)^2H_0\\ &\qquad +60(m_3)^2\Omega_0-87q_0=0,\\ &78(m_3)^2X_0-68(m_3)^2H_0+55(m_3)^2\Omega_0-63q_0=0,\\ &54(m_3)^2X_0+18(m_3)^2H_0-5[11-9(m_3)^2]\Omega_0+54q_0=0,\\ &6X_0-6(m_3)^2H_0+4\Omega_0-3q_0=0. \end{aligned} \quad (245)$$

A simple, rather tedious manipulation shows that four of the constants (say X_0, Y_0, H_0, and Ω_0) are proportional to the fifth (q_0), as well as polynomials of $(m_3)^2$:

$$\begin{aligned} X_0 &= -\frac{1}{4}P_1 q_0, \\ H_0 &= -\frac{3}{4(m_3)^2}P_2 q_0, \\ Y_0 &= \frac{1}{19}\left[\frac{2}{5}\left[5+9(m_3)^2\right]X_0 - \frac{3}{10}(m_3)^2 H_0 \right. \\ &\qquad \left. +3(m_3)^2\Omega_0 - \frac{87}{20}q_0\right], \end{aligned} \tag{246}$$

where

$$\begin{aligned} P_1 &\equiv \frac{M_1\left[1+3(m_3)^2\right]^{-1}}{\left[450(m_3)^4+1434(m_3)^2-935\right]}, \\ P_2 &\equiv \frac{M_2\left[1+3(m_3)^2\right]^{-1}}{\left[450(m_3)^4+1434(m_3)^2-935\right]}, \\ P_3 &\equiv \frac{\left[39(m_3)^6+5(m_3)^4-5(m_3)^2+8\right]}{\left[450(m_3)^4+1434(m_3)^2-935\right]} \\ M_1 &\equiv 4455(m_3)^8+5328(m_3)^6+1524(m_3)^4 \\ &\quad -3583(m_3)^2+598, \\ M_2 &\equiv 900(m_3)^8+2523(m_3)^6+3591(m_3)^4 \\ &\quad -3465(m_3)^2-240. \end{aligned}$$

The following condition on $(m_3)^2$ then results:

$$\begin{aligned} &24[15-11(m_3)^2]X_0+22(m_3)^2 H_0 \\ &\quad -220(m_3)^2\Omega_0+319q_0=0, \end{aligned}$$

which, upon substitution of X_0, H_0, Ω_0 and q_0, reduces to a fifth-order equation on $(m_3)^2$ that is satisfied for at least one positive value of $(m_3)^2$ namely, $(m_3)^2 \approx 2.9252$; this proves the consistency of the Ansatz. The solution for a vectorial perturbation, for a Kasner background with an isotropy plane, is then written in the form

$$\begin{aligned} q(t) &= q_0 t^{-1}, \\ X(t) &= X_0, \\ Y(t) &= Y_0 t^{-1}, \\ H(t) &= H_0 t^{-5/3}, \\ \Omega(t) &= \Omega_0 t^{-5/3}, \end{aligned} \tag{247}$$

with constants X_0, Y_0, H_0, and Ω_0 expressed in terms of q_0 by (246).

In contrast with the tensorial case as we shall see, (248) shows that the only existing solution is stable, even if the decrease is not fast.

3.3.5 Tensorial Perturbations

This section discusses gravitational waves in the Kasner background. The minimal closed set $\mathcal{M}$ now reduces to the four tensorial variables:

$$\mathcal{M} = \{X_{\alpha\beta}, Y_{\alpha\beta}, H_{\alpha\beta}, \pi_{\alpha\beta}\}. \tag{248}$$

However, from causal thermodynamics, the following relation between the shear and the anisotropic pressure can be obtained:

$$\tau(\pi_{ij})^{\cdot} + \pi_{ij} = \xi\sigma_{ij}, \tag{249}$$

where τ and ξ are the relaxation viscosity parameters, respectively.

We will repeat here the choice made for the FLRW case and consider τ as negligible. The viscosity will also be taken roughly as a constant,[11] which reduces (249) to the form

$$\pi_{ij} = \xi\sigma_{ij}. \tag{250}$$

This result poses a problem after perturbation, for it would then imply that the anisotropic pressure—a "good" variable in the sense of Stewart (cf. Novello [118]), one that is zero in the background and therefore gauge-independent—can be written in terms of the shear, which, in our case, is nonzero and (in the sense of Stewart) coordinate dependent, a dependence that characterizes a "bad" variable. To solve this apparent dilemma, we can take the viscosity as a "good" variable itself (since it is zero in the background and therefore gauge independent as well). However, the viscosity is a scalar quantity and, as such, not defined for tensorial perturbations. The solution to the problem, then, is to consider the viscosity itself as zero, i.e., write

$$(\delta\pi_{ij}) = \xi(\delta\sigma_{ij}) = 0, \tag{251}$$

after perturbation, so that the consistency of the dynamical system is maintained.

This further reduces the set $\mathcal{M}$:

$$\mathcal{M} = \{X_{\alpha\beta}, Y_{\alpha\beta}, H_{\alpha\beta}\}. \tag{252}$$

We can, at this point, expand the perturbed quantities in $\mathcal{M}$ in terms of the tensorial basis $\hat{U}^{\mu}{}_{\nu}$ as follows:

$$\begin{aligned} (\delta X^i{}_j) &= \sum_{(n)} X(t)^{(n)}\hat{U}_{(n)}{}^i{}_j, \\ (\delta Y^i{}_j) &= \sum_{(n)} Y(t)^{(n)}\hat{U}_{(n)}{}^i{}_j, \\ (\delta H^i{}_j) &= \sum_{(n)} H(t)^{(n)}\hat{U}^*{}_{(n)}{}^i{}_j. \end{aligned} \tag{253}$$

Since we will deal with linear equations, we can henceforth suppress the summation and the extra indices and deal

[11] In nonequilibrium, thermodynamics, both τ and ξ, are functions of the system equilibrium variables, such as the density ρ and the temperature T.

with each component (n) separately. The same reasoning will be applied to the vectorial and to the scalar cases as well.

With (253), a perturbed dynamical system can then be written. Starting from the original QM equations and rewriting them in terms of the above-defined new variables, we can exhibit the perturbed dynamical system for the tensorial case as follows:

$$h^{\mu}_{\alpha}h^{\beta}_{\nu}(\delta X^{\alpha}{}_{\beta})^{\cdot}+\frac{5}{3}\theta(\delta X^{\mu}{}_{\nu})+\sigma^{\mu}_{\alpha}(\delta X^{\alpha}{}_{\nu})+\sigma^{\alpha}_{\nu}(\delta X^{\mu}{}_{\alpha}) -\frac{2}{3}h^{\mu}_{\nu}\sigma^{\alpha}_{\beta}(\delta X^{\beta}{}_{\alpha})-(\delta Y^{\mu}{}_{\nu}) +\frac{1}{2}\eta^{\mu\gamma}{}_{\alpha}\beta V_{\gamma}h^{\lambda}_{\nu}(\delta H^{\alpha}_{\lambda})_{;\beta}+\frac{1}{2}\eta_{\nu}{}^{\gamma}{}_{\alpha}{}^{\beta}V_{\gamma}h^{\mu\lambda}(\delta H^{\alpha}_{\lambda})_{;\beta}=0, \tag{254}$$

$$h^{\mu}_{\alpha}h^{\beta}_{\nu}(\delta Y^{\alpha}{}_{\beta})^{\cdot}+2\theta(\delta Y^{\mu}{}_{\nu})-\frac{3}{2}\sigma^{\mu}_{\alpha}(\delta Y^{\alpha}{}_{\nu})-\frac{3}{2}\sigma^{\alpha}_{\nu}(\delta Y^{\mu}{}_{\alpha}) +h^{\mu}_{\nu}\sigma^{\alpha}_{\beta}(\delta Y^{\beta}{}_{\alpha})+\frac{3}{2}E^{\mu}_{\alpha}(\delta X^{\alpha}{}_{\nu}) +\frac{3}{2}E^{\alpha}_{\nu}(\delta X^{\mu}{}_{\alpha})-h^{\mu}_{\nu}E^{\alpha}_{\beta}(\delta X^{\beta}{}_{\alpha})+\frac{1}{2}\theta\eta^{\mu\gamma\alpha\beta}V_{\gamma}h_{\nu\lambda}(\delta H^{\lambda}_{\alpha})_{;\beta} +\frac{1}{2}\theta\eta_{\nu}{}^{\gamma\alpha\beta}V_{\gamma}h^{\mu}_{\lambda}(\delta H^{\lambda}_{\alpha})_{;\beta} +\frac{3}{4}\eta^{\mu\gamma\alpha\beta}V_{\gamma}\sigma_{\nu\lambda}(\delta H^{\lambda}_{\alpha})_{;\beta}+\frac{3}{4}\eta_{\nu}{}^{\gamma\alpha\beta}V_{\gamma}\sigma^{\mu}_{\lambda}(\delta H^{\lambda}_{\alpha})_{;\beta} +\frac{3}{4}\eta^{\gamma\lambda\alpha\beta}V_{\lambda}\sigma^{\mu}_{\gamma}h_{\nu\tau}(\delta H^{\tau}_{\alpha})_{;\beta}+ +\frac{3}{4}\eta^{\gamma\lambda\alpha\beta}V_{\lambda}\sigma_{\gamma\nu}h^{\mu}_{\tau}(\delta H^{\tau}_{\alpha})_{;\beta}-h^{\mu}_{\nu}\eta^{\gamma\lambda\alpha\beta}V_{\lambda}\sigma_{\gamma\tau}(\delta H^{\tau}_{\alpha})_{;\beta}=0, \tag{255}$$

$$2\sigma^{\alpha}_{\beta}\left(\delta X^{\beta}_{\alpha}\right)=0, \tag{256}$$

$$h^{\mu}_{\alpha}h^{\beta}_{\nu}(\delta H^{\alpha}{}_{\beta})^{\cdot}+\frac{4}{3}\theta(\delta H^{\mu}{}_{\nu})-\frac{1}{2}\sigma^{\mu}_{\alpha}(\delta H^{\alpha}{}_{\nu}) -\frac{1}{2}\sigma^{\alpha}_{\nu}(\delta H^{\mu}{}_{\alpha})+\eta^{\mu\alpha\gamma}{}_{\epsilon}\eta_{\nu}{}^{\beta\lambda\tau}V_{\gamma}V_{\lambda}\sigma_{\alpha\beta}(\delta H^{\epsilon}_{\tau}) +\frac{1}{3}\theta\eta^{\mu\alpha\gamma}{}_{\epsilon}\eta_{\nu}{}^{\beta\lambda\tau}V_{\gamma}V_{\lambda}\sigma_{\alpha\beta}(\delta H^{\epsilon}_{\tau}) -\frac{1}{2}\eta^{\mu\gamma\alpha\beta}V_{\gamma}h_{\nu\lambda}\left(\delta H^{\lambda}_{\alpha}\right)_{;\beta}-\frac{1}{2}\eta_{\nu}{}^{\gamma\alpha\beta}V_{\gamma}h^{\mu}_{\lambda}\left(\delta H^{\lambda}_{\alpha}\right)_{;\beta} \tag{257}$$

$$h^{\alpha}_{\beta}h^{\mu\nu}\left(\delta X^{\beta}{}_{\mu}\right)_{;\nu}+\eta^{\alpha\beta\mu\nu}V_{\beta}\sigma_{\mu\gamma}\left(\delta H^{\gamma}_{\nu}\right)=0, \tag{258}$$

and

$$h^{\alpha}_{\beta}h^{\mu\nu}\left(\delta H^{\beta}{}_{\mu}\right)_{;\nu}+\eta^{\alpha\beta\mu\nu}V_{\beta}\sigma_{\mu\gamma}\left(\delta X^{\gamma}_{\nu}\right)=0. \tag{259}$$

This dynamic system has to be decomposed in terms of the tensor basis and then solved for the specific Kasner solution. However, it is immediately seen (upon writing the perturbed terms on the tensorial basis) that the following restriction on the background must be accepted in order that the basis can be factored out from the equations

$$p_1=p_2, \tag{260}$$

which implies the existence of an *isotropy plane* in the Kasner original, nonperturbed background. There are two such solutions, named Kasner and Milne:

(1) *Kasner solution*

$$p_1 = p_2 = 2/3,\quad p_3 = -1/3;$$

(2) *Milne solution*

$$p_1 = p_2 = 0,\quad p_3 = 1.$$

Here, we will only consider the Kasner solution, since the Milne case has already been analyzed by Novello [123].

Additional choices will be made on the tensorial basis which, while not constituting a material change on $\hat{U}^{\mu}{}_{\nu}$, will simplify the algebraic steps towards a final closed dynamical system:

$$\hat{U}^{3}{}_{1}=\hat{U}^{3}{}_{2}=0,$$

whereupon we write

$$\hat{U}^{\mu}{}_{\nu}=\begin{pmatrix}\alpha & 0 & 0\\ 0 & -\alpha & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{261}$$

and

$$\hat{U}^{*\mu}{}_{\nu}=-\frac{i}{2}\eta^{0123}k_3\begin{pmatrix}0 & -2t^{2p_2}\alpha & 0\\ -2t^{2p_1}\alpha & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{262}$$

Proceeding to analyze the dynamical system, we find that (256) is identically satisfied, since

$$\left(\sigma^{\alpha}{}_{\beta}\hat{U}^{\beta}{}_{\alpha}\right)=0, \tag{263}$$

and similar show that Eqs. (258) and (259) are also identically valid.

The remaining equations can then be rewritten in the form

$$\begin{aligned}\dot{X}\ &+\frac{7}{3}\theta X-Y+k^2H=0,\\ \dot{Y}\ &+\theta Y+\frac{2}{3}\theta^2X+2\theta k^2H=0,\\ \dot{H}\ &+\frac{2}{3}\theta H-t^{-4/3}X=0,\end{aligned} \tag{264}$$

which constitutes a closed dynamical system in the variables (X, Y, H). This result is analogous to the ones obtained in the FLRW case, but in this case, the system can be completely solved by using the relations

$$\theta = t^{-1},$$
$$k^2 = t^{-2/3}(k_3)^2, \quad k_3 \equiv const,$$

and considering a simple form for the desired solution, in terms of powers of t,

$$\begin{aligned} X(t) &= X_0 t^x, \\ Y(t) &= Y_0 t^y, \\ H(t) &= H_0 t^w, \end{aligned} \tag{265}$$

with X_0, Y_0, H_0, x, y and w as constants to be determined.

We substitute this Ansatz in (265) to obtain the following equations:

$$\begin{aligned} &(3x+7)X_0 t^{(x-1)} - 3Y_0 t^y + 3(k_3)^2 H_0 t^{(w-2/3)} = 0, \\ &2X_0 t^{(x-1)} - 3(y+1)Y_0 t^y + 6(k_3)^2 H_0 t^{(w-2/3)} = 0, \\ &3X_0 t^{(x-1)} - (3w+2)H_0 t^{(w-2/3)} = 0. \end{aligned} \tag{266}$$

It is immediately seen that the only nontrivial solutions to (266) satisfy the conditions

$$y = x - 1 = w - \frac{2}{3} \Rightarrow w = x - \frac{1}{3}, \tag{267}$$

and the dynamical system reduces to the following conditions on the triad (X_0, Y_0, H_0) and the constant $(k_3)^2$:

$$\begin{aligned} &(3x+7)X_0 - 3Y_0 + 3(k_3)^2 H_0 = 0, \\ &2X_0 - 3(y+1)Y_0 + 6(k_3)^2 H_0 = 0, \\ &3X_0 - (3w+2)H_0 = 0. \end{aligned} \tag{268}$$

These conditions then determine X_0 and Y_0 in terms of H_0 and the exponents x, y, w:

$$\begin{aligned} X_0 &= \frac{(3w+2)}{3} H_0, \\ Y_0 &= \frac{2}{3}\frac{(x+2)(3w+2)}{(y+3)} H_0, \end{aligned} \tag{269}$$

which, upon employing (267) and (269) in the Ansatz (266), gives

$$\begin{aligned} X(t) &= \frac{(3w+2)}{3} t^{1/3} H(t), \\ Y(t) &= \frac{2}{3}\frac{(3w+2)^2}{(3w+7)} t^{-2/3} H(t), \\ H(t) &= H_0 t^w. \end{aligned} \tag{270}$$

We also determine the constant $(k_3)^2$, which is expressed in terms of x, y, w by the relation

$$(k_3)^2 = -\frac{1}{9}(3w+2)^2. \tag{271}$$

If we take the arbitrary constant H_0 as positive, an analysis of the stability of the above solutions above, (270), yields Table 3. The tabulated results show that tensorial perturbations of a Kasner background may present—upon a choice of the exponent w—the same kind of instability present in the Friedman-Lemaître-Robertson-Walker spacetime. This instability is rather gradual, not catastrophic as in the Einstein model. This is a reasonable development, since we are not interested in eliminating the background anisotropy under a perturbation.

Table 3 Stability-analysis results for tensorial perturbations

Value for w	$X(t)$	$Y(t)$	$H(t)$
$w < -2/3$	Stable	Stable	Stable
$w = -2/3$	Null	Null	Stable
$-2/3 < w < -1/3$	Stable	Stable	Stable
$w = -1/3$	Constant	Stable	Stable
$-1/3 < w < 0$	Unstable	Stable	Stable
$w = 0$	Unstable	Stable	Constant
$0 < w < 2/3$	Unstable	Stable	Unstable
$w = 2/3$	Unstable	Constant	Unstable
$w > 2/3$	Unstable	Unstable	Unstable

3.4 Friedman Universe: Scalar Perturbations

In the case of the spatially homogeneous, isotropic FLRW cosmological model, the vanishing of the Weyl conformal tensor suggests that the QM approach is more useful. Therefore, the variation of Weyl conformal tensor $\delta W_{\alpha\beta\mu\nu}$ is the basic quantity to be considered, since $\delta W_{\alpha\beta\mu\nu}$ is, with no doubt, a true perturbation, which cannot be achieved by a coordinate transformation. This solves ab initio the aforementioned gauge problem.

From a technical standpoint, instead of considering tensorial quantities, one should restrain oneself to scalar ones. There are two ways to implement this:

- Expand the relevant quantities on a complete basis of functions (e.g., the spherical harmonics basis)
- Analyze the invariant geometric quantities one can construct from $g_{\mu\nu}$ and its derivatives in the Riemannian background structure, that is, examine the 14 Debever invariants

Either way, we shall see that the net result is that there is a set of perturbed quantities that can be divided into "good" quantities [i.e., ones whose unperturbed counterparts have zero value in the background and, consequently, Stewart's lemma (cf. Stewart [148]) guarantees that the associated perturbed quantity is really a gauge-independent one] and "bad" ones (whose background values are nonzero). One should therefore limit the analysis to the "good" ones.

The same kind of behavior occurs for the geometrical structure of the model for both the kinematic and dynamic quantities of matter. Therefore, the "good" quantities, the set of variables with which we work, should then be chosen from the particular scalars that come from these three structures: geometric, kinematic and dynamic. Does that mean that the present approach effectively avoids the gauge problem?

To answer this question affirmatively, one should be able to exhibit a set of "good" variables in such a way that its corresponding dynamics is closed. That is, if we call $\mathcal{M}_{[A]}$ the set of these variables, Einstein's equations should provide the dynamics of each element of $\mathcal{M}_{[A]}$, depending only on the background evolution quantities (and, eventually, on other elements of $\mathcal{M}_{[A]}$). This would exhaust the perturbation problem. We shall show that this is indeed the case.

What we learn from this discussion is that one should understand the gauge problem not as a basic difficulty of perturbation theory, but as a simple matter of asking a bad question.[12] One could imagine—which has been used a number of times in the literature (Hawking [65], Olson [128] and Mukhanov [101])—that for the FRW cosmology, the perturbations of its main characteristics (the energy density $\delta\rho$, the scalar of curvature δR and the Hubble expansion factor $\delta\theta$) would be natural candidates to be considered as basic for the perturbation scheme. However, these are not "good" scalars, since they are nonzero in the background.[13] We shall see in the next sections which scalars replace these ones.

Consider the FRW geometry written in the standard Gaussian coordinate system as in (64). The three-dimensional geometry has constant curvature and thus the corresponding Riemannian tensor $\hat{R}_{ijkl}$ can be written in the form

$$\hat{R}_{ijkl} = -\epsilon\gamma_{ijkl}.$$

where $\gamma_{ijkl} \equiv \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}$ was defined in (72). For the moment, it is necessary to distinguish covariant derivatives in the four-dimensional space-time by the symbol (;) and the three-dimensional derivatives by ($_{\|}$).

Since the original Lifshitz paper, it has been shown to be useful to develop all perturbed quantities on the spherical harmonics basis. Since we are limiting ourselves to irrotational perturbations, it suffices for our purposes to take into account only the scalar $Q(x^k)$ (with $\dot{Q} = 0$) and its derived vector and tensor quantities. We have thus

$$\begin{aligned} Q_i &\equiv Q_{,i}, \\ Q_{ij} &\equiv Q_{,i;j}, \end{aligned} \tag{272}$$

where the scalar Q obeys the following eigenvalue equation, defined in the three-dimensional background space:

$$\hat{\nabla}^2 Q = mQ, \tag{273}$$

[12] Let us point out that some of the gauge-dependent terms are particularly relevant, $\delta\rho$ here included.

[13] However, as we shall see soon, we can construct associated "good" quantities in terms of these scalars.

where m is the wave number of the scalar eigenfunction, with

$$m = \begin{cases} q^2+1, \ 0<q<\infty, & \epsilon = 1 \text{ (open)}, \\ q, \qquad\ \ 0<q<\infty, & \epsilon = 0 \text{ (plane)}, \\ n^2-1, \ n=1,2,\ldots, & \epsilon = -1 \text{ (closed)}, \end{cases} \tag{274}$$

and

$$\hat{\nabla}^2 Q \equiv \gamma^{ik} Q_{,i\|k} = \gamma^{ik} Q_{,i;k}, \tag{275}$$

where the symbol $\hat{\nabla}^2$ denotes the three-dimensional Laplace-Beltrami operator. The traceless operator $\bar{Q}_{ij}$ is defined as

$$\bar{Q}_{ij} = \frac{1}{m} Q_{ij} - \frac{1}{3} Q\gamma_{ij}, \tag{276}$$

and the divergence of $\bar{Q}_{ij}$ is given by

$$\bar{Q}^{ik}_{\|k} = 2\left(\frac{1}{3} + \frac{\epsilon}{m}\right) Q^i. \tag{277}$$

We remark that Q is a three-dimensional object; therefore, indices are raised with γ^{ij}, the three-space metric.

Debever [40] presented the complete 14 algebraically independent invariants constructed with the curvature tensor. Considering that we are using an dimensionless metric tensor, we can classify them with respect to dimensionality as follows:

Dimensionality	Invariants
L^{-2}	I_5
L^{-4}	I_1, I_3, I_6
L^{-6}	$I_2, I_4, I_7, I_9, I_{12}$
L^{-8}	I_8, I_{10}, I_{13}
L^{-10}	I_{11}, I_{14}

The expressions for these invariants are:

$$\begin{aligned}
I_1 &= W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu}, & I_8 &= C_{\alpha\beta} C^{\beta\mu} C_{\mu\lambda} C^{\alpha\lambda}, \\
I_2 &= W_{\alpha\beta}{}^{\rho\sigma} W_{\rho\sigma}{}^{\mu\nu} W_{\mu\nu}{}^{\alpha\beta}, & I_9 &= C_{\mu\nu} D^{\mu\nu}, \\
I_3 &= W^{\alpha\beta\mu\nu}\, {}^*W_{\alpha\beta\mu\nu}, & I_{10} &= D_{\mu\nu} D^{\mu\nu}, \\
I_4 &= W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu}\, {}^*W_{\mu\nu\alpha\beta}, & I_{11} &= C_{\alpha\beta} D^{\beta\mu} D_\mu{}^\alpha, \\
I_5 &= R, & I_{12} &= \tilde{D}_{\mu\nu} C^{\mu\nu}, \\
I_6 &= C_{\mu\nu} C^{\mu\nu}, & I_{13} &= \tilde{D}_{\mu\nu} D^{\mu\nu}, \\
I_7 &= C_{\alpha\beta} C^{\beta\mu} C_\mu{}^\alpha, & I_{14} &= \tilde{D}_{\mu\nu} \tilde{D}^{\nu\alpha} C^\mu{}_\alpha.
\end{aligned}$$

where we have used the following definitions:

$$\begin{aligned} C_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}, \\ D_{\mu\nu} &\equiv W_{\mu\alpha\nu\beta} C^{\alpha\beta}, \\ \tilde{D}_{\mu\nu} &\equiv {}^*W_{\mu\alpha\nu\beta} C^{\alpha\beta}. \end{aligned} \tag{278}$$

3.4.1 Fundamental Perturbations of the FLRW Universe

As we observed previously, a complete examination of perturbation theory should naturally include the analysis of the evolution of the Debever metric invariants associated with the FLRW geometry.

The only invariants of the FLRW geometry that are not identically zero are given by the expression

$$I_5 = (1-3\lambda)\rho,$$
$$I_6 = \frac{3}{4}(1+\lambda)^2\rho^2,$$
$$I_7 = -\frac{3}{8}(1+\lambda)^3\rho^3,$$
$$I_8 = \frac{21}{64}(1+\lambda)^4\rho^4,$$

where we have used Einstein's equations and the stress-energy tensor is that of a perfect fluid.

If we restrict ourselves to linear perturbation theory, the only invariants which have nonidentically zero linear perturbation terms are I_5, I_6, I_7, I_8, I_9 and I_{12}. Among these, the first four are nonzero in the background and the latter two are zero, since the geometry is conformally flat. This could lead to the conclusion that I_9 and I_{12} are the "good" scalars to be examined. However, direct calculation shows that the latter two invariants have zero linear perturbation. Indeed, it follows from FLRW geometry that the perturbation of I_9 reduces to

$$\delta I_9 = C^{\mu\nu} C^{\alpha\beta} \delta W_{\mu\alpha\nu\beta}.$$

Given that the Weyl tensor is trace-free, we see that the above quantity vanishes identically. Of course, this result depends on the source of the background geometry being given by a perfect fluid. In effect, we have in this case

$$\delta I_9 = (\rho+p)^2\left(V^\mu V^\nu - \frac{1}{4}g^{\mu\nu}\right) \times \left(V^\alpha V^\beta - \frac{1}{4}g^{\alpha\beta}\right)\delta W_{\mu\alpha\nu\beta},$$

which is zero.

Likewise δI_{12}, given by the equality

$$\delta I_{12} = C^{\mu\nu} C^{\alpha\beta} \delta\ {}^*W_{\mu\alpha\nu\beta},$$

also vanishes.

The corresponding perturbations for the remaining invariants are given by the expression

$$\delta I_5 = (1-3\lambda)\delta\rho,$$
$$\delta I_6 = \frac{3}{2}(1+\lambda)^2\rho\delta\rho,$$
$$\delta I_7 = -\frac{9}{8}(1+\lambda)^3\rho^2\delta\rho,$$
$$\delta I_8 = \frac{21}{16}(1+\lambda)^4\rho^3\delta\rho.$$

It follows from these results that the perturbations of these quantities are algebraically related.[14] Besides, since all these scalars have a nonzero background value, they do not belong to the minimum set of good quantities that we are searching.

Corresponding difficulties occur for the standard kinematical and dynamical variables, that is, the expansion parameter θ and the density of energy ρ suffer from the same disease.

This is, thus, the bad choice for the basic variables that we should avoid. Let us now turn our attention to the good variables that should be considered as the fundamental ones.

Geometric Perturbations From the previous section, it follows that

$$\sqrt{\delta E_{ij}\delta E^{ij}},$$

is the only quantity that characterizes without ambiguity a true perturbation of the Debever invariants.[15] We therefore only need to consider the perturbed E_{ij}, since, as we shall see, any other metric quantity does not belong to the "good" basic nucleus needed for complete knowledge of the true perturbations. We then set the expansion of this tensor on the spherical-harmonic basis

$$\delta E_{ij} = E(t)\bar{Q}_{ij}(x^k). \tag{279}$$

Thus $E(t)$ is the geometric quantity whose dynamics we are looking for.

Kinematical Perturbations We restrict our considerations to linear perturbation terms only. The normalization of the four-velocity yields that the variation in the time component of the perturbed velocity is related to the variation in the (0-0) component of the metric tensor, that is:

$$\delta V_0 = \frac{1}{2}\delta g_{00}. \tag{280}$$

The corresponding contravariant quantities are related as follows:

$$\delta V^0 = \frac{1}{2}\delta g^{00} = -\delta V_0. \tag{281}$$

The expansion of the perturbations of the four-velocity on the spherical harmonic basis reads[16]

$$\delta V_0 = \frac{1}{2}\beta(t)Q(x^i) + \frac{1}{2}Y(t),$$
$$\delta V_k = V(t)Q_k(x^i). \tag{282}$$

[14]One can write these invariants in a pure geometrical way without using Einstein's equations. This does not modify our argument.

[15]This is a consequence of the vanishing of the perturbation of the magnetic part of Weyl tensor(cf.above).

[16]The vorticity is of course zero, since we are limiting ourselves to the irrotational case.

For the acceleration, we set

$$\delta a_k = \Psi(t) Q_k(x^i). \tag{283}$$

For the shear,

$$\delta\sigma_{ij} = \Sigma(t)\bar{Q}_{ij}(x^k), \tag{284}$$

and for the expansion,

$$\delta\theta = H(t)Q(x^i) + Z(t), \tag{285}$$

where $Y(t)$ and $Z(t)$ are homogeneous terms that are not true perturbations.

Since we are limiting ourselves to analyzing true perturbed quantities, $\Sigma(t)$ is the only important kinematical variable whose dynamics we need to examine; the other gauge-invariant quantity Ψ is a function of Σ (and E), as we shall see (β is just a matter of choosing the coordinate system).

Matter Perturbation Since we are considering a background geometry in which a state equation relates the pressure and the energy density, i.e., $p = \lambda\rho$, we will consider the standard procedure accepting the preservation of this state equation under arbitrary perturbations. Besides, our frame is such that there is no heat flux. Thus, the general form of the perturbed energy-momentum tensor is given by the equation

$$\delta T_{\mu\nu} = (1+\lambda)\delta(\rho V_\mu V_\nu) - \lambda\delta(\rho g_{\mu\nu}) + \delta\Pi_{\mu\nu}. \tag{286}$$

We write $\delta\rho$ in terms of the scalar basis as follows:

$$\delta\rho = N(t)Q(x^i) + \mu(t), \tag{287}$$

where the homogeneous term $\mu(t)$ is not a true perturbation.[17]

According to causal thermodynamics, the evolution equation of the anisotropic pressure is related to the shear through the relation

$$\tau\dot{\Pi}_{ij} + \Pi_{ij} = \xi\sigma_{ij}, \tag{288}$$

where τ is the relaxation parameter and ξ is the viscosity parameter, as we saw in the previous section.

For simplicity of the present treatment, we will limit ourselves to the case in which τ can be neglected and ξ is a constant;[18] (288) then yields the result

$$\Pi_{ij} = \xi\sigma_{ij}, \tag{289}$$

[17] We will set $Y = Z = \mu = 0$, since these homogeneous terms are just a matter of choosing the coordinate system. Nevertheless, we are not interested in examining such pure gauge quantities as Y, Z and μ.

[18] In the general case, ξ and τ are functions of the equilibrium variables (for instance, ρ and the temperature T) and, since both variations $\delta\Pi_{ij}$ and $\delta\sigma_{ij}$ are expanded in terms of the traceless tensor $\hat{Q}_{ij}$, it follows that the above relation does not restrain the kind of fluid we are examining. However, if we consider ξ as time-dependent, the quantity $\delta\Pi_{ij}$ must be included in the fundamental set $\mathcal{M}_{[A]}$.

and the associated perturbed equation is:

$$\delta\Pi_{ij} = \xi\delta\sigma_{ij}. \tag{290}$$

Following the same reasoning as before, $\delta\Pi_{ij}$ is the matter quantity that should enter the complete system of differential equations that describes the perturbation evolution. One should also be interested in the dynamics of $\delta\rho$, although it is not a fundamental part of the basic system of equations. We will examine its evolution later on.

The "good" set $\mathcal{M}_{[A]}$ has therefore three elements: δE_{ij}, $\delta\sigma_{ij}$ and $\delta\Pi_{ij}$. But, since $\delta\Pi_{ij}$ is written in terms of $\delta\sigma_{ij}$, the set $\mathcal{M}_{[A]}$ that will be considered reduces to:

$$\mathcal{M}_{[A]} = \{\delta E_{ij}, \delta\sigma_{ij}\}.$$

So much for definitions. Let us turn to the analysis of the dynamics.

3.4.2 Dynamics

In this section, we will show that $E(t)$ and $\Sigma(t)$ constitute the fundamental pair of variables that determine the dynamics for the perturbed FRW geometry, that is, $\mathcal{M}_{[A]} = \{E(t), \Sigma(t)\}$ is the minimal closed set of observables in the perturbation theory of FRW that characterizes and completely determines the spectrum of perturbations. Indeed, the evolution equations for these two quantities (which come from Einstein's equations) generate a dynamical system only involving E and Σ (and background quantities), which when solved contains all the necessary information for a complete description of all remaining perturbed quantities of the FRW geometry. This conclusion seems not to have been noticed in the past.

Our discussion will be limited to examining the perturbed quantities that are relevant for complete knowledge of the system. These are the quasi-Maxwellian equations of gravitation and the evolution equations for the kinematical quantities. Vishniac [73] and Novello [121] have presented and analyzed this system of equations.

The Perturbed Equation for the Shear The perturbed equation for the shear, (106), reads

$$h_\alpha{}^\mu h_\beta{}^\nu\,(\delta\sigma_{\mu\nu})^{\cdot} + \frac{2}{3}\theta\,\delta\sigma_{\alpha\beta} + \frac{1}{3}h_{\alpha\beta}\,\delta a^\lambda{}_{;}$$

$$\lambda - \frac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu[\delta a_{\mu;\nu} + \delta a_{\nu;\mu}] = \delta M_{\alpha\beta}, \tag{291}$$

where

$$M_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu}V^\mu V^\nu - \frac{1}{3}R_{\mu\nu}V^\mu V^\nu h_{\alpha\beta}. \tag{292}$$

The above-developed spherical harmonics expansion and (290) reduce (291) to the form

$$\dot{\Sigma} = -E - \frac{1}{2}\xi\Sigma + m\Psi. \tag{293}$$

Perturbed Equation for E_{ij} The perturbed equation for the electric part of the Weyl tensor is (101). Using the above-derived spherical harmonics expansion and (290), we find that

$$\dot{E} = -\frac{(1+\lambda)}{2}\rho\,\Sigma - \left(\frac{\Theta}{3}+\frac{\xi}{2}\right)E - \frac{\xi}{2}\left(\frac{\xi}{2}+\frac{\Theta}{3}\right)\Sigma + \frac{m}{2}\xi\,\Psi. \tag{294}$$

This suggests that E and Σ be considered as canonically conjugate variables. We shall see later on that this is indeed the case.

Equations (293) and (294) contain three variables: E, Σ and Ψ. We will now show that the conservation law for matter can be exploited to eliminate Ψ in all cases, except when $(1+\lambda)=0$. We will return to this particular (vacuum) case in a later section.

The proof is the following. Projecting the conservation equation of the energy-momentum tensor in the three-space, we have that

$$T^{\mu\nu}{}_{;\nu}h_\mu{}^\lambda = 0. \tag{295}$$

Using the perturbed quantities, (295) yields the expression

$$(1+\lambda)\rho\delta a_k - \lambda(\delta\rho)_{,k} + \lambda\dot{\rho}\delta V_k + \delta\pi_k{}^i{}_{;i} = 0. \tag{296}$$

The decomposition on the spherical-harmonics basis then yields the relation

$$(1+\lambda)\rho\Psi = \lambda[N - \dot{\rho}V] + 2\xi\left(\frac{1}{3}+\frac{\epsilon}{m}\right)a^{-2}\Sigma. \tag{297}$$

A remarkable result then follows: the right-hand side of (297) can be expressed in terms of the variables E and Σ only, since we are analyzing here the case $(1+\lambda)\neq 0$. Indeed, from the equation for the divergence of the electric tensor—see (529)—we see that

$$N - \dot{\rho}V = \left(1+\frac{3\epsilon}{m}\right)\xi\,\Sigma a^{-2} - 2\left(1+\frac{3\epsilon}{m}\right)a^{-2}E. \tag{298}$$

Combining these two equations, we find that Ψ is given in terms of the background quantities and the basic perturbed terms E and Σ:

$$(1+\lambda)\rho\Psi = 2\left(1+\frac{3\epsilon}{m}\right)a^{-2}\left[-\lambda E + \frac{1}{2}\lambda\xi\,\Sigma + \frac{1}{3}\xi\,\Sigma\right]. \tag{299}$$

Thus, the whole set of perturbed equations reduces, for the variables E and Σ, to a time-dependent dynamical system:

$$\dot{\Sigma} = F_1(\Sigma, E),$$
$$\dot{E} = F_2(\Sigma, E), \tag{300}$$

with

$$F_1 \equiv -E - \frac{1}{2}\xi\,\Sigma + m\Psi,$$

and

$$F_2 \equiv -\left(\frac{1}{3}\theta + \frac{1}{2}\xi\right)E - \left(\frac{1}{4}\xi^2 + \frac{(1+\lambda)}{2}\rho + \frac{1}{6}\xi\theta\right)\Sigma + \frac{m}{2}\xi\Psi,$$

in which Ψ is given in terms of E and Σ by (299).

3.4.3 Comparison with Previous Gauge-Invariant Variables

FLRW cosmology is characterized by the homogeneity of the fundamental variables that specify its kinematics (the expansion factor θ), its dynamics (the energy density ρ) and its associated geometry (the scalar of curvature R). This means that these three quantities depend only on the global time t, characterized by the hyper-surfaces of homogeneity. We can therefore define in a trivial way three-tensor associated quantities, which vanish in this geometry and look for its corresponding nonidentically vanishing perturbation. The simplest way to do this is just to let U be a homogeneous variable (in the present case, it can be any one of the quantities ρ, θ, or R), that is $U = U(t)$. Then, we use the three-gradient operator $\hat{\nabla}_\mu$ defined by

$$\hat{\nabla}_\mu \equiv h_\mu{}^\lambda\nabla_\lambda, \tag{301}$$

to produce the desired associated variable

$$U_\mu = \hat{\nabla}_\mu U. \tag{302}$$

Ellis and Bruni [50] have discussed these quantities, analyzed their associated evolution and compared the results with the standard perturbative approach. In the present section, we will relate these variables to our fundamental ones. We shall see that under the conditions of our analysis[19] these quantities are functionals of our basic variables (E and Σ) and the background variables.

The Matter Variable χ_i It proves convenient to define the fractional gradient of the energy density χ_α as

$$\chi_\alpha \equiv \frac{1}{\rho}\hat{\nabla}_\alpha\rho. \tag{303}$$

The quantity χ_α is but a combination of the acceleration and the divergence of the anisotropic stress. Indeed, in the frame without heat flux, from (296), it follows

$$\delta\chi_i = \frac{(1+\lambda)}{\lambda}\delta a_i + \frac{1}{\lambda\rho}\delta\pi_i{}^\beta{}_{;\beta}, \tag{304}$$

[19]Recall that our discussion is restricted to irrotational perturbations. Our results are therefore simpler. Our method, however, is free from this restriction and generic cases can equally well be obtained along the same lines.

From what we have learned, it follows that this quantity can be reduced to a functional of the basic quantities of perturbation, that is Σ and E, yielding

$$\delta\chi_i = -2\left(1+\frac{3\epsilon}{m}\right)\frac{1}{\rho a^2}\left(E-\frac{\xi}{2}\Sigma\right)Q_i. \tag{305}$$

The Kinematical Variable η_i The only nonvanishing quantity in the kinematics of the cosmic background fluid is the (Hubble) expansion factor θ. This allows us to define η_α as follows:

$$\eta_\alpha = h_\alpha{}^\beta \theta_{,\beta}. \tag{306}$$

The constraint relation (4) then relates this quantity to the basic ones:

$$\delta\eta_i = -\frac{\Sigma}{a^2}\left(1+\frac{3\epsilon}{m}\right)Q_i. \tag{307}$$

We can choose the scalar of curvature R, which like ρ and θ depends only on the cosmical time t, to be the U-geometrical variable. However, it seems more appealing to use a combination τ involving R, ρ and θ defined by the equality

$$\tau = R + (1+3\lambda)\rho - \frac{2}{3}\theta^2. \tag{308}$$

In the unperturbed FLRW background, τ is defined in terms of the curvature scalar of the three-dimensional space and the scale factor $a(t)$:

$$\frac{\hat{R}}{a^2}.$$

We therefore define the new associated variable τ_α as

$$\tau_\alpha = h_\alpha{}^\beta \tau_{,\beta}. \tag{309}$$

The variable τ_α vanishes in the background. Its perturbation can be written in terms of the previous variations, since Einstein's equations give

$$\tau = 2\left(\rho - \frac{1}{3}\theta^2\right).$$

Therefore, without any information loss, we can limit all our analysis to the fundamental variables. Nevertheless, for the sake of completeness, we exhibit the evolution equations for a few gauge-dependent variables.

Perturbed Equations for ρ and θ From (111), using the decomposition of the perturbed energy density on the scalar basis [defined by (287)], we obtain the following equation of evolution for $\delta\rho$:

$$\dot{N} - \frac{1}{2}\beta\dot{\rho} + (1+\lambda)\theta N + (1+\lambda)\rho H = 0. \tag{310}$$

Applying the same procedure to the perturbed Raychaudhuri (108) and decomposing on the scalar basis (285), we obtain the equation

$$\dot{H} - \frac{1}{2}\beta\dot{\theta} + \frac{2}{3}\theta H + \frac{m}{a^2}\Psi + \frac{(1+3\lambda)}{2}N = 0. \tag{311}$$

To solve these two equations, we need to fix the gauge $\beta(t)$ and use the E and Σ obtained from the fundamental closed system (300). All the remaining geometrical and kinematical quantities can be likewise obtained. This completely exhausts our analysis of the irrotational perturbations of the FLRW universe.

3.4.4 Singular Case $(1+\lambda)=0$: the Perturbations of the de Sitter Universe

We have seen that all the system of reduction to the variables Σ and E was based on the possibility of writing the acceleration in terms of E and Σ. This was possible in all cases, except for $(1+\lambda)=0$. Although no fluid is known with such a negative pressure, the fact that the vacuum admits this interpretation has led to the identification of the cosmological constant with this fluid. We therefore examine this case in the same way as was done in the previous sections.

In contrast with all previously studied cases, perturbations of this fluid must necessarily contain contributions from the heat flux or the anisotropic pressure. Indeed, if we take both of these quantities as vanishing, then the set of perturbed equations implies that all equations are trivially satisfied, since all perturbative quantities vanish, except for the cases where $\delta p = \overline{\lambda}\delta\rho$, with $\overline{\lambda}=0$ and $\lambda+1=0$. We will analyze these cases below.

When $\delta p = \overline{\lambda}\delta\rho$, for $\overline{\lambda}=0$, the system is stable. Indeed, we obtain for the electric part of Weyl tensor, in the case that θ is constant in the background, the following expression:

$$E(t) = E_0 e^{-\frac{\theta}{3}t}.$$

The other case of interest is the one in which the condition $(1+\lambda)=0$ is preserved throughout the perturbation. From (310), since $\dot{\rho}=0$ and $(1+\lambda)=0$, it follows that temporal variation of the energy density exists only if we take into account the perturbed fluid with heat flux. We then write

$$q_i = q(t)Q_i(x^k).$$

Equation (111) gives the expression

$$\dot{N} = \frac{m}{a^2}q. \tag{312}$$

The projected conserved equation yields the expression [see (112)]

$$\dot{q} + \theta q + N = \frac{2\xi}{3a^2}\left(1+\frac{3\epsilon}{m}\right)\Sigma. \tag{313}$$

The evolution equation for the electric part of Weyl tensor shows that

$$\dot{\mathcal{S}} + \frac{\theta}{3}\mathcal{S} = -\frac{m}{2}q, \tag{314}$$

where we have used the definition

$$\mathcal{S} \equiv E - \frac{1}{2}\xi\Sigma.$$

Finally, from the equation for the divergence of E_{ij}, we have the constraint

$$\frac{2}{a^2}\left(1 + \frac{3\epsilon}{m}\right)\mathcal{S} = -(N + \theta q). \tag{315}$$

The evolution equation for the shear determines the acceleration Ψ. Equations (313)–(315) thus constitute a complete system for the variables E, Σ and q. This completes the general explicitly gauge-invariant scheme that we have presented here, including the singular case $(1+\lambda) = 0$. Before moving to another topic, just as an additional comment, we note that it would be interesting to consider the perturbation scheme in the framework of the Lanczos potential. This will be done in a later section.

3.4.5 Hamiltonian Treatment of the Scalar Solution

The examination of the perturbations in FLRW cosmology, which we analyzed above, admits a Hamiltonian formulation, which we now consider (cf. Grishchuk [63]). In this vein, the variables E and Σ, analyzed in the previous section, are the ones that must be employed. From the evolution equations for Σ and E (300), it follows that they are not canonically conjugated for arbitrary background geometries.

The natural step would be to define canonically conjugated variables Q and P as a linear functional of Σ and E as:[20]

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \alpha & \eta \\ \delta & \beta \end{bmatrix} \begin{bmatrix} \Sigma \\ E \end{bmatrix}. \tag{316}$$

Functionals of the background geometry should be expected to appear in the construction of the canonical variables in the functions α, β, η and δ. This matrix is univocally defined up to canonical transformations, a freedom we can use to choose η and δ as zero; we shall use this choice to simplify our analysis.

The Hamiltonian $\mathcal{H}$ providing the dynamics of the pair (Q, P) is obtained from the evolution equations of E and Σ, (300). The condition for the existence of such a Hamiltonian is given by the equation

$$\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} - \xi - \frac{1}{3}\theta + \frac{2m\xi}{3(1+\lambda)\rho a^2}\left(1 + \frac{3\epsilon}{m}\right) = 0. \tag{317}$$

[20] The quantity Q in this subsection should not be confused with the previous scalar basis.

It then follows that the Hamiltonian that provides the dynamics of our problem takes the form

$$\mathcal{H} = \frac{h_1}{2}Q^2 + \frac{h_2}{2}P^2 + 2h_3 P Q, \tag{318}$$

where h_1, h_2 and h_3 are defined as

$$h_1 \equiv \frac{\beta}{\alpha}\left[\frac{(1+\lambda)}{2}\rho + \frac{\xi}{2}\left(\frac{\xi}{2} + \frac{\theta}{3}\right) - \frac{m\xi^2}{(1+\lambda)\rho a^2}\left(1 + \frac{3\epsilon}{m}\right)\left(\frac{\lambda}{2} + \frac{1}{3}\right)\right], \tag{319}$$

$$h_2 \equiv -\frac{\alpha}{\beta}\left[1 + \frac{2m\lambda}{(1+\lambda)\rho a^2}\left(1 + \frac{3\epsilon}{m}\right)\right], \tag{320}$$

and

$$h_3 \equiv \frac{\theta}{3} - \frac{\dot{\beta}}{\beta} + \frac{\xi}{4}\left[1 + \frac{2m\lambda}{(1+\lambda)\rho a^2}\left(1 + \frac{3\epsilon}{m}\right)\right]. \tag{321}$$

Let us consider the case in which $\xi = 0$, that is, there is no anisotropic pressure. The case in which ξ is nonvanishing presents some interesting peculiarities, which will be left to a forthcoming section.

We will choose $\beta = a$ and take α as given by (317). We then define the canonical variables Q and P by setting

$$Q = \Sigma,$$
$$P = aE.$$

It then follows that $\mathcal{H}$ is given by

$$\mathcal{H} = -\Delta^2(t)P^2 + \gamma^2(t)Q^2, \tag{322}$$

where $\gamma(t)$ and $\Delta(t)$ are given in terms of the energy density of the background ρ, the scale factor $a(t)$ and the wave number m as:

$$\gamma^2(t) \equiv \left[\frac{(1+\lambda)}{4}\right]\rho a,$$
$$\Delta^2(t) \equiv \frac{1}{2a}\left[1 + \frac{2m\lambda}{(1+\lambda)\rho a^2}\left(1 + \frac{3\epsilon}{m}\right)\right]. \tag{323}$$

Let us make two comments here: first, the system is not conservative (that is, $\dot{\mathcal{H}}$ is not zero) because the ground state of this theory ($Q = P = 0$) corresponds not to the Minkowski flat space-time but to the FLRW expanding universe. The second remark is that the same statement applies to the nonpositivity of the Hamiltonian, also a consequence of the nonvanishing curvature of the fundamental state. The system under study is not closed; momentum and energy can therefore be pumped from the background.

We notice that the Hamiltonian structure obtained in terms of the variables E and Σ is completely gauge-invariant and, as such, deserves additional analysis, which we will present elsewhere. Here, we only want to exhibit an example in which this pumping effect can be easily recognized. To this end, we apply the Hamiltonian treatment to a static model of the universe.

3.4.6 Fierz-Lanczos Potential

As remarked before, perturbations of conformally flat space times do not require[21] complete knowledge of all components of the perturbed metric tensor $\delta g_{\mu\nu}$, although they certainly need to take into account the Weyl conformal tensor, which contains all the necessary observable information (namely, δE_{ij} and δH_{ij}).

The tensor $W_{\alpha\beta\mu\nu}$ can be expressed in terms of the three-index Fierz-Lanczos potential tensor—see Fierz [53] and Lanczos [86]—which will be denoted by $L_{\alpha\beta\mu}$ and deserves careful analysis. Indeed, one could consider $\delta L_{\alpha\beta\mu}$ as the good object for studying linear perturbation theory, since, as we shall see, it combines both $\delta\Sigma_{ij}$ and δa_k (which are alternative variables to describe δE_{ij}).

Before going into the perturbation-related details, let us summarize a few definitions and properties of $L_{\alpha\beta\mu}$, since the literature has very few papers on this matter.[22]

Basic Properties In any four-dimensional Riemannian geometry, there is a three-index tensor $L_{\alpha\beta\mu}$ with the following symmetries:

$$L_{\alpha\beta\mu} + L_{\beta\alpha\mu} = 0 \tag{324}$$

and

$$L_{\alpha\beta\mu} + L_{\beta\mu\alpha} + L_{\mu\alpha\beta} = 0. \tag{325}$$

With such a $L_{\alpha\beta\mu}$, we may write the Weyl tensor in the form of a homogeneous expression in the potential expression, that is

$$\begin{aligned} W_{\alpha\beta\mu\nu} &= L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]} \\ &\quad + \frac{1}{2}\left[L_{(\alpha\nu)}g_{\beta\mu} + L_{(\beta\mu)}g_{\alpha\nu} \right. \\ &\qquad \left. - L_{(\alpha\mu)}g_{\beta\nu} - L_{(\beta\nu)}g_{\alpha\mu}\right] \\ &\quad + \frac{2}{3}L^{\sigma\lambda}{}_{\sigma;\lambda}g_{\alpha\beta\mu\nu}, \end{aligned} \tag{326}$$

where

$$L_{\alpha\mu} \equiv L_{\alpha}{}^{\sigma}{}_{\mu;\sigma} - L_{\alpha;\mu}$$

and

$$L_{\alpha} \equiv L_{\alpha}{}^{\sigma}{}_{\sigma}.$$

Due to the above symmetry properties, (324) and (325), the Lanczos tensor has 20 degrees of freedom. Since the Weyl tensor has only ten independent components, we see that gauge symmetry is involved. This gauge symmetry can be separated into two classes:

$$\Delta^{(1)}L_{\alpha\beta\mu} = M_{\alpha}g_{\beta\mu} - M_{\beta}g_{\alpha\mu}, \tag{327}$$

and

$$\begin{aligned} \Delta^{(2)}L_{\alpha\beta\mu} &= W_{\alpha\beta;\mu} - \frac{1}{2}W_{\mu\alpha;\beta} + \frac{1}{2}W_{\mu\beta;\alpha} \\ &\quad + \frac{1}{2}g_{\mu\alpha}W_{\beta}{}^{\lambda}{}_{;\lambda} - \frac{1}{2}g_{\mu\beta}W_{\alpha}{}^{\lambda}{}_{;\lambda}, \end{aligned} \tag{328}$$

where the vector M_{α} and the antisymmetric tensor $W_{\alpha\beta}$ are arbitrary quantities.

Lanczos Tensor for FLRW Geometry The conformal flatness of the Friedman-Lematre-Robertson-Walker geometry implies that the associated Lanczos potential is but a gauge. That is, we can write the Lanczos potential for FRW geometry in the form

$$\begin{aligned} L_{\alpha\beta\mu} &= N_{\alpha}g_{\beta\mu} - N_{\beta}g_{\alpha\mu} + F_{\alpha\beta;\mu} - \frac{1}{2}F_{\mu\alpha;\beta} \\ &\quad + \frac{1}{2}F_{\mu\beta;\alpha} + \frac{1}{2}g_{\mu\alpha}F_{\beta}{}^{\lambda}{}_{;\lambda} - \frac{1}{2}g_{\mu\beta}F_{\mu}{}^{\lambda}{}_{;\lambda}, \end{aligned} \tag{329}$$

for the arbitrary vector N_{α} and the antisymmetric tensor $F_{\alpha\beta}$.

Perturbed Fierz-Lanczos Tensor In the case examined in this paper (irrotational perturbations), the perturbed Weyl tensor reduces to the form

$$\delta W_{\alpha\beta\mu\nu} = (\eta_{\alpha\beta\gamma\varepsilon}\eta_{\mu\nu\lambda\rho} - g_{\alpha\beta\gamma\varepsilon}g_{\mu\nu\lambda\rho})V^{\gamma}V^{\lambda}\delta E^{\varepsilon\rho}, \tag{330}$$

since the magnetic part of Weyl tensor remains zero in this case.

It then follows that the perturbed electric tensor is given in terms of Lanczos potential by the expression

$$\begin{aligned} -\delta E_{ij} &= \delta L_{0i[0;j]} + \delta L_{0j[0;i]} - \frac{1}{2}\delta L_{(00)}\gamma_{ij} \\ &\quad - \frac{1}{2}\delta L_{(ij)} + \frac{2}{3}\delta L^{\sigma\lambda}{}_{\sigma;\lambda}\gamma_{ij}. \end{aligned} \tag{331}$$

Although the $L_{\alpha\beta\mu}$ tensor is not uniquely well defined (since it has the above-discussed gauge freedom), we can use certain theorems—see Novello [126] and López Bonilla [4]—that enable one to write $L_{\alpha\beta\mu}$ in terms of the associated kinematic quantities of a given congruence of curves in the associated Riemannian manifold. Following these theorems and choosing the case of irrotational perturbed matter, it follows that $\delta L_{\alpha\beta\mu}$ (the perturbed tensor of FLRW background) is given by the equation

$$\delta L_{\alpha\beta\mu} = \delta\sigma_{\mu[\alpha}V_{\beta]} + F(t)\delta a_{[\alpha}V_{\beta]}V_{\mu}, \tag{332}$$

[21] The above-mentioned gauge problem has been widely discussed in the literature (see [124] and references therein).

[22] This tensor was introduced in the 1930s to provide, much as the symmetric tensor $\varphi_{\mu\nu}$ does—in a more often used approach—an alternative description of spin-2 field in the Minkowski background. In the 1960s, Lanczos rediscovered it, without recognizing he was dealing with the same object, as a Lagrange multiplier used to obtain the Bianchi identities in the context of Einstein's General Relativity. However, only recently (cf. Novello [114, 115]) a complete analysis of Fierz-Lanczos object was undertaken and it was discovered that its generic (Fierz) version describes not only one, but two spin-2 fields. The restriction to a single spin-2 field is usually called the Lanczos tensor. We will limit all our considerations to this restricted quantity.

where

$$F(t) = 1 - \frac{1}{m}\frac{\Sigma}{\Psi}\left(\frac{2}{3}\theta + \frac{1}{2}\xi\right). \tag{333}$$

In other words, the only nonidentically zero components of $\delta L_{\alpha\beta\mu}$ are:

$$\delta L_{0k0} = -F(t)\Psi Q_k, \tag{334}$$

and

$$\delta L_{0ij} = -\Sigma(t)\bar{Q}_{ij}, \tag{335}$$

which coincides with the previous results.

From what we have learned in the previous section, we can conclude that this is not a univocal expression, that is, (334) and (335) are obtained with a specific gauge choice.

Let us apply the above gauge transformation to the present case. In the first gauge, (327), we decompose vector M_α in the spatial harmonics (scalar and vector):

$$M_0 = M^{(1)}(t)Q(x), \tag{336}$$

and

$$M_i = M^{(2)}(t)Q_i(x), \tag{337}$$

and in the second gauge, (328), we have that

$$W_{0i} = W^{(1)}(t)Q_i(x), \tag{338}$$

and

$$W_{ij} = -\frac{1}{a^2}\varepsilon_{ijk}W^{(2)}(t)Q^k(x). \tag{339}$$

To sum up, asking what the Lanczos tensor is for the perturbed FLRW geometry is one of the questions, like the one concerning the perturbed tensor $\delta g_{\mu\nu}$, that should be avoided, since this quantity is gauge-dependent. As remarked before, a good question is to ask what the perturbation of the Weyl tensor is. This was precisely the motivation of the previous section.

3.5 Friedman Universe: Vectorial Perturbations

As discussed in the previous section, we will use the perturbation formalism in Einstein's theory of gravitation, which is based on gauge-independent and evident physically meaningful quantities, such as the vorticity, shear, electric and magnetic parts of the conformal Weyl tensor and others.

In the scalar case, the convention has been simplified in order to simplify the resulting system of dynamical equations. For the vectorial and tensorial cases, however, we feel that the convention set by Hawking [65] is more adequate. Therefore, we will present it here.

The metric of the background is given in the standard Gaussian form, which defines a class of privileged observers $V^\alpha = \delta^\alpha_0$. The projector $h_{\mu\nu}$ defines, in the three-dimensional space orthogonal to V^α, the three-dimensional quantities with the symbol $(\widehat{\ })$. Thus, $\hat{X}_\alpha \equiv h_\alpha{}^\beta X_\beta$ denotes a projection into the three-geometry. Following the same reasoning, we define the operator $\hat{\nabla}_\alpha$ as the covariant derivative in the three-geometry. The relation between the three-dimensional Laplacian ($\hat{\nabla}^2$) and the four-dimensional one is given as follows:

$$\hat{\nabla}^2\,\hat{X}_\alpha = \left(\frac{\theta}{3}\right)^2\,\hat{X}_\alpha + h^\beta_\alpha\,\nabla^2\,\hat{X}_\beta.$$

We then introduce the fundamental harmonic basis of the functions projected onto the three-surface

$$\{Q(x)\}\,,\ \{\hat{P}_\alpha(x)\},\ \{\hat{U}_{\alpha\beta}(x)\}. \tag{340}$$

In this section, we are interested in the vector basis $\hat{P}_\alpha(x)$, which is defined by the following relations:

$$\begin{aligned}\hat{P}_\mu\,V^\mu &= 0,\\ \dot{\hat{P}}^\mu &= 0,\\ \hat{\nabla}^\alpha\,\hat{P}_\alpha &= 0,\\ \hat{\nabla}^2\,\hat{P}_\alpha &= \frac{m}{a^2}\,\hat{P}_\alpha,\end{aligned} \tag{341}$$

where the eigenvalue (again denoted by m, although this eigenvalue and the one in the scalar basis have no relation at all) is given by

$$m = \begin{cases} q^2+2,\ 0<q<\infty, & \epsilon = +1\ \text{(open)},\\ q, \qquad\ \ 0<q<\infty, & \epsilon = \ \ 0\ \text{(plane)},\\ n^2-2,\ n=2,3,\ldots, & \epsilon = -1\ \text{(closed)}.\end{cases} \tag{342}$$

From this basis, it is possible to derive a pseudo-vector and a tensor:

$$\begin{aligned}\hat{P}^{*\alpha} &\equiv \eta^{\alpha\beta\mu\nu}\,V_\beta\,\hat{P}_{\mu\nu},\\ \hat{P}_{\alpha\beta} &\equiv \hat{\nabla}_\beta\,\hat{P}_\alpha,\\ \hat{P}^*_{\alpha\beta} &\equiv \hat{\nabla}_\beta\,\hat{P}^*_\alpha,\end{aligned} \tag{343}$$

suited to develop pseudo-vectors and tensors.

The following vectorial relations are useful in obtaining the dynamical equations:

$$\begin{aligned}
\dot{\hat{P}}_{(\alpha\beta)} &= -\frac{1}{3}\,\theta\,\hat{P}_{(\alpha\beta)},\\
\dot{\hat{P}}^{*}_{(\alpha\beta)} &= -\frac{2}{3}\,\theta\,\hat{P}^{*}_{(\alpha\beta)},\\
\dot{\hat{P}}^{*}_{\alpha} &= -\frac{1}{3}\,\theta\,\hat{P}^{*}_{\alpha},\\
\hat{\nabla}^{\beta}\,\hat{P}_{(\alpha\beta)} &= \frac{1}{A^2}\,(m+2\epsilon)\,\hat{P}_{\alpha},\\
\hat{\nabla}^{\beta}\,\hat{P}^{*}_{(\alpha\beta)} &= \frac{1}{A^2}\,(m+2\epsilon)\,\hat{P}^{*}_{\alpha},\\
\hat{\nabla}^{2}\,\hat{P}^{*}_{\alpha} &= \frac{m}{a^2}\,\hat{P}^{*}_{\alpha},\\
\eta^{\alpha\beta\gamma\varepsilon}\,V_{\beta}\,\hat{P}^{*}_{\gamma;\varepsilon} &= \frac{1}{a^2}\,(m-2\epsilon)\,\hat{P}^{\alpha},\\
h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\eta_{\mu}{}^{\lambda\gamma\tau}\,V_{\tau}\,\hat{\nabla}_{\gamma}\,\hat{P}_{(\nu\lambda)} &= h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\hat{P}^{*}_{\mu\nu},\\
h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\eta_{\mu}{}^{\lambda\gamma\tau}\,V_{\tau}\,\hat{\nabla}_{\gamma}\,\hat{P}_{[\nu\lambda]} &= -h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\hat{P}^{*}_{\mu\nu}.
\end{aligned}\tag{344}$$

where we have used the constraint relation,

$$-\frac{\epsilon}{a^2} - \frac{1}{3}\,\rho + \left(\frac{\theta}{3}\right)^2 = 0,\tag{345}$$

valid in the FLRW background.

The following auxiliary relations are also useful:

$$\begin{aligned}
\dot{\theta} &= -\frac{1}{3}\,\theta^2 - \frac{1}{2}\,(\rho+3p),\\
\dot{\rho} &= -\theta\,(\rho+p).
\end{aligned}\tag{346}$$

With the above basis, we are able to expand any *good* perturbed quantity (again denoted by δX, where X is any quantity associated to the matter content, kinematics and geometry) as follows:

$$\begin{aligned}
\delta\omega_{\alpha} &= \Omega(t)\,\hat{P}^{*}_{\alpha},\\
\delta q_{\alpha} &= q(t)\,\hat{P}_{\alpha},\\
\delta a_{\alpha} &= \Psi(t)\,\hat{P}_{\alpha},\\
\delta V_{\alpha} &= V(t)\,\hat{P}_{\alpha},\\
\delta\sigma_{\alpha\beta} &= \Sigma(t)\,\hat{P}_{(\alpha\beta)},\\
\delta H_{\alpha\beta} &= H(t)\,\hat{P}^{*}_{(\alpha\beta)},\\
\delta E_{\alpha\beta} &= E(t)\,\hat{P}_{\alpha\beta},\\
\delta\pi_{\alpha\beta} &= \pi(t)\,\hat{P}_{(\alpha\beta)}.
\end{aligned}\tag{347}$$

3.5.1 Dynamics

In order to derive simpler equations, we will again consider the thermodynamic equation,

$$\tau\,\dot{\Pi}_{\alpha\beta} + \Pi_{\alpha\beta} = \xi\,\sigma_{\alpha\beta},\tag{348}$$

in the limit of small relaxation time τ (adiabatic approximation) and constant viscosity coefficient ξ to obtain the approximate form

$$\delta\Pi_{\alpha\beta} = \xi\,\delta\sigma_{\alpha\beta} \;\rightsquigarrow\; \Pi = \xi\,\Sigma.\tag{349}$$

The vorticity can be written in terms of the three-velocity as

$$\delta\omega_{\alpha} = -\frac{1}{2}\,\delta V_{\alpha} \;\rightsquigarrow\; V = -2\,\Omega.\tag{350}$$

We will denote by $(\chi_r,\ \tilde{\Phi}_s)$ the fundamental dynamical and constraint equations, respectively. Introducing (348) and (349)–(350) into the perturbed quasi-Maxwellian (101)–(112) and making use of (341)–(344), we get

$$\dot{E} - \frac{1}{2}\xi\,\dot{\Sigma} + \frac{2}{3}\theta\,E + \frac{1}{2}(\rho+p)\,\Sigma + \frac{1}{2a^2}\,(m-2\epsilon)\,H + \frac{1}{4}q = 0,\tag{351a}$$

$$\dot{\Sigma} + \left(\frac{\theta}{3} + \frac{\xi}{2}\right)\Sigma + E - \frac{1}{2}\,\Psi = 0,\tag{351b}$$

$$\dot{\Omega} + \frac{1}{3}\theta\,\Omega + \frac{1}{2}\,\Psi = 0,\tag{351c}$$

$$\dot{H} + \frac{1}{3}\theta\,H - \frac{1}{2}\,E - \frac{1}{4}\xi\,\Sigma = 0,\tag{351d}$$

$$\dot{q} + \frac{4}{3}\theta\,q + \frac{1}{a^2}\,(m+2\epsilon)\,\xi\,\Sigma + 2\dot{p}\,\Omega + (\rho+p)\,\Psi = 0,\tag{351e}$$

and

$$\Sigma + \Omega + 2H = 0,\tag{352a}$$

$$\frac{1}{a^2}\,(m+2\epsilon)\,E - \frac{1}{2a^2}\,(m+2\epsilon)\,\xi\,\Sigma + \frac{2}{3}\,\theta\,(\rho+p)\,\Omega - \frac{1}{3}\theta\,q = 0,\tag{352b}$$

$$\frac{1}{a^2}\,(m+2\epsilon)\,H - (\rho+p)\,\Omega + \frac{1}{2}\,q = 0,\tag{352c}$$

$$\frac{1}{a^2}\,(m+2\epsilon)\,\Sigma + \left\{\frac{1}{a^2}\,(m-2\epsilon) + 4\left(\frac{\theta}{3}\right)^2 + \frac{2}{3}\,(\rho+p)\right\}\Omega - q = 0.\tag{352d}$$

With help of (345)–(346), it can be easily shown that constraint (352d) is unessential, since it is written in terms of (352a) and (352c). We can write the constraint (352b) in a simpler form as

$$E - \frac{1}{2}\,\xi\,\Sigma + \frac{2}{3}\,\theta\,H = 0.\tag{353}$$

The fundamental differential system is now written as

$$\dot{E} - \frac{1}{2}\,\xi\,\dot{\Sigma} + \frac{2}{3}\,\theta\,E + \frac{1}{2}\,(\rho+p)\,\Sigma + \frac{1}{2a^2}\,(m-2\epsilon)\,H + \frac{1}{4}\,q = 0, \tag{354a}$$

$$\dot{\Sigma} + \left(\frac{\theta}{3} + \frac{\xi}{2}\right)\,\Sigma + E - \frac{1}{2}\,\Psi = 0, \tag{354b}$$

$$\dot{\Omega} + \frac{1}{3}\,\theta\,\Omega + \frac{1}{2}\,\Psi = 0, \tag{354c}$$

$$\dot{H} + \frac{1}{3}\,\theta\,H - \frac{1}{2}\,E - \frac{1}{4}\,\xi\,\Sigma = 0, \tag{354d}$$

$$\dot{q} + \frac{4}{3}\,\theta\,q + \frac{1}{a^2}\,(m+2\epsilon)\,\xi\,\Sigma - 2\dot{p}\,\Omega + (\rho+p)\,\Psi = 0, \tag{354e}$$

and

$$\Sigma + \Omega + 2H = 0, \tag{355a}$$

$$E - \frac{1}{2}\,\xi\,\Sigma + \frac{2}{3}\,\theta\,H = 0, \tag{355b}$$

$$\frac{1}{A^2}\,(m+2\epsilon)\,H - (\rho+p)\,\Omega + \frac{1}{2}\,q = 0. \tag{355c}$$

It could be argued that the acceleration Ψ should be eliminated from the dynamical system via the definition

$$a_\alpha = \dot{V}_\alpha = V_{\alpha;\beta}\,V^\beta.$$

If this is done, we obtain the expression

$$\Psi\,\hat{P}_\alpha = \left(\dot{V} + \frac{\theta}{3}\,V\right)\,\hat{P}_\alpha - \delta\Gamma^0_{0\alpha}.$$

However, it is easily proven (see Novello [122]) that

$$\delta\Gamma^0_{0\alpha} = \frac{1}{2}\,(\delta\,g_{00})_{,\alpha} = (\delta\,V_0)_{,\alpha},$$

which is zero in the vector basis.

Then, assisted by (350), we have the following relation:

$$\Psi = -2\,\dot{\Omega} - \frac{2}{3}\,\theta\,\Omega,$$

which is precisely the dynamical (354c).

The variable Ψ can then be eliminated only at the expense of some degrees of freedom. This way we get physically motivated (i.e., by observation) algebraic relations between acceleration and other selected variables. We will restrict ourselves here to the three cases that follow.

The first possible choice is to admit a shear-free model for the cosmological perturbation. There is no shear in this case and hence the anisotropic pressure vanishes, too. We will therefore refer to this case as "isotropic" henceforth in this section. Equation (354b) then becomes

$$\Psi = 2\,E. \tag{356}$$

The second possibility is to admit that no vorticity should be taken into account. As it has long been known, a non-vanishing vorticity usually brings difficulties related to causality violation. Our motivation for this choice is therefore to eliminate the main source of causality breakdown. In this case, we have

$$\Omega = 0,$$

and (354c) then yields the result

$$\Psi = 0. \tag{357}$$

Another possibility is to require the physical source of curvature to be a Stokesian fluid, so that the energy flux (heat flux, in this case) vanishes. Even though we can always set this quantity to zero by a suitable choice of observers, it actually represents a true restriction, for our equations are written in such a way that the observer cannot be changed, that is, we have already fixed the observer by imposing the particle flux to vanish. Now, (354e) yields

$$\Psi = -\frac{1}{a^2}\,(m+2\epsilon)\,\frac{\xi}{(\rho+p)}\,\Sigma + 2\,\lambda\,\theta\,\Omega, \tag{358}$$

with

$$(\rho+p) = (1+\lambda)\,\rho \neq 0 \qquad \lambda \equiv const,$$

a relation that eliminates Ψ for all but the de Sitter background. A subsequent section will study the dynamics and Hamiltonian treatment of each of the three possibilities.

3.5.2 Permanence of Constraints

Since we obtained a constrained differential system, given by (354a–354e) and (355a–355c), it is useful to consider whether constraints are or are not automatically preserved. If we differentiate (355a–355c) and inserts into the relations (345)–(346) in the results, we come directly to the relation

$$\begin{aligned}
\dot{\Phi}_1 &= \chi_2 + \chi_3 + 2\chi_4 - \frac{\theta}{3}\,\Phi_1,\\
\dot{\Phi}_2 &= \chi_1 - \frac{2}{3}\,\theta\,\chi_4 - \frac{1}{2}\,(\rho+p)\,\Phi_1 - \frac{\theta}{3}\,\Phi_2 + \frac{1}{2}\,\Phi_3\\
&\quad - \left[\frac{-\epsilon}{a^2} + \left(\frac{\theta}{3}\right)^2 - \frac{1}{3}\,\rho\right]\,(\Omega+2H),\\
\dot{\Phi}_3 &= -(\rho+p)\,\chi_3 + \frac{1}{a^2}\,(m+2\epsilon)\,\chi_4 + \frac{1}{2}\,\chi_5\\
&\quad - \frac{1}{2a^2}\,(m+2\epsilon)\,\Phi_2 - \frac{2}{3}\,\theta\,\Phi_3.
\end{aligned} \tag{359}$$

where Φ_i, $i = 1, 2, 3$ are the constraints, (355a–355c) and χ_j, $j = 1, ..., 5$ are the evolution (354a–354e). Thus, it follows that no secondary constraint[23] (SC) appears in the case of vector perturbations. This should be expected, since this result reflects the fact the dynamical equivalence between our basic (quasi-Maxwellian) equations and Einstein's field equations, which are complete.

3.5.3 Hamiltonian Treatment of the Vectorial Solution

If we keep all degrees of freedom, as we mentioned before, the simplest solution for (354a–354e)–(355a–355c) is then to let Ψ be a small arbitrary function of the background, i.e., $\Psi = \Psi(t)$, which can also be parameterized by the perturbation wavelength m.

The constraints can now be used to eliminate three of the five variables and the most suitable pair for this solution is (Σ, Ω). The resulting free dynamics is

$$\dot{\Sigma} = -\left(\frac{2}{3}\theta + \xi\right)\Sigma - \frac{\theta}{3}\Omega + \frac{1}{2}\Psi,$$
$$\dot{\Omega} = -\frac{\theta}{3}\Omega - \frac{1}{2}\Psi, \qquad (360)$$

directly integrable to yield the results

$$\Sigma(t) = a^{-2}(t)\,e^{-\xi t}\left\{C_1 + \int_{(H_0^{-1}+c_0)}^{t} a^2(t')\,e^{\xi t'} \times\left[-\frac{\theta(t')}{3}\Omega(t') + \frac{1}{2}\Psi(t')\right]dt'\right\},$$

$$\Omega(t) = a^{-1}(t)\left\{C_2 - \int_{(H_0^{-1}+c_0)}^{t} \frac{1}{2}a(t')\,\Psi(t')\,dt'\right\}, \qquad (361)$$

where H_0 is the Hubble parameter and c_0 a positive integration constant.

Solution (361) can be thought of as a particular case of an arbitrary linear relation[24] between Ψ and the fundamental variables,

$$\Psi = y(t)\,Q + z(t)\,P + g(t), \qquad (362)$$

where

$$y(t) := \frac{\partial\Psi}{\partial Q},\; z(t) := \frac{\partial\Psi}{\partial P}$$

and (Q, P) is a pair of canonical variables (as we shall see) that describe the vector perturbations, given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \Sigma \\ \Omega \end{pmatrix}, \quad \begin{pmatrix} \Sigma \\ \Omega \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} \qquad (363)$$

where $\Delta \equiv \alpha\delta - \beta\gamma \neq 0$.

The above choice of variables is motivated by traditional results of perturbations assuming a perfect fluid law; within this assumption, both the vorticity and the shear are essential variables: none of them may vanish, or else all the system turns out to be trivial (see Goode [59]). In the more general case, this result is inapplicable.

Differentiating (363) and using the dynamics in (360), we find the following dynamics in terms of (Q, P):

$$\dot{Q} = \left[\dot{\alpha} - \left(\frac{2}{3}\theta + \xi\right)\alpha\right]\Sigma + \left[\dot{\beta} - \frac{\theta}{3}(\alpha + \beta)\right]\Omega + \frac{1}{2}(\alpha - \beta)\Psi,$$
$$\dot{P} = \left[\dot{\gamma} - \left(\frac{2}{3}\theta + \xi\right)\gamma\right]\Sigma + \left[\dot{\delta} - \frac{\theta}{3}(\gamma + \delta)\right]\Omega + \frac{1}{2}(\gamma - \delta)\Psi. \qquad (364)$$

To ensure that we are actually working with canonically conjugated variables, we write the Hamiltonian constraint

$$\begin{aligned}\Phi :&= \Delta\left(\frac{\partial\dot{Q}}{\partial Q} + \frac{\partial\dot{P}}{\partial P}\right) \\ &= \left[\dot{\alpha} - \left(\frac{2}{3}\theta + \xi\right)\alpha\right]\delta - \left[\dot{\beta} - \frac{\theta}{3}(\alpha + \beta)\right]\gamma \\ &\quad - \left[\dot{\gamma} - \left(\frac{2}{3}\theta + \xi\right)\gamma\right]\beta \\ &\quad \left[\dot{\delta} - \frac{\theta}{3}(\gamma + \delta)\right]\alpha + \frac{\Delta}{2}(\alpha - \beta)\,y \\ &\quad + \frac{\Delta}{2}(\gamma - \delta)\,z \\ &= \dot{\Delta} - (\theta + \xi)\Delta + \frac{1}{2}\left[(\alpha - \beta)\,y + (\gamma - \delta)\,z\right]\Delta,\end{aligned} \qquad (365)$$

and set the solution of $\Phi = 0$ in the form

$$\Delta = \frac{a^3(t)}{a_0}e^{\xi t}, \alpha = \delta = \Delta^{1/2}, \qquad (366)$$

with $a_0 = const$.

Equation (366) will indeed be a solution if

$$(\alpha - \beta)\,y + (\gamma - \delta)\,z = 0$$

[23] Terminology due to Bergmann relative to Dirac's work [41] on constrained systems.

[24] Linearity required to preserve the coherence with our basic assumption of linear-perturbations approximation. For the understanding of the physical meaning of this relation, see the examples in Section 3.5.1.

holds, which leads to the following three possibilities:

$$\begin{array}{lllll} i) & y \neq 0,\ \forall z & \rightarrow f \equiv \frac{z}{y}, & \beta = \alpha(1-f), & \gamma = 0; \\ ii) & y = 0,\ z \neq 0 & \rightarrow & \beta = 0, & \gamma = \alpha; \\ iii) & y = z = 0 & \rightarrow & \beta = 0, & \gamma = 0. \end{array} \tag{367}$$

For the first case, the dynamics results in

$$\dot{Q} = \left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta - \xi + \frac{\alpha}{2}z\right) Q + \left[-\frac{\theta}{3}f + \xi(1-f) - \dot{f} - \frac{\alpha}{2}f z\right] P + \frac{\alpha}{2} f g, \tag{368}$$

$$\dot{P} = -\frac{\alpha}{2} y Q + \left(\frac{\dot{\alpha}}{\alpha} - \frac{\theta}{3} - \frac{\alpha}{2}z\right) P - \frac{\alpha}{2} g, \tag{369}$$

described by the Hamiltonian

$$\mathcal{H}(Q,P) = \frac{\alpha}{4}Q^2 + \frac{1}{2}\left[-\frac{\theta}{3}f + \xi(1-f) - \dot{f} - \frac{\alpha}{2}f z\right]P^2 - \left(\frac{\dot{\alpha}}{\alpha} - \frac{\theta}{3} - \frac{\alpha}{2}z\right) Q P + \frac{\alpha}{2}(Q + f g P). \tag{370}$$

For the second case, the dynamics becomes

$$\dot{Q} = \left(\frac{\dot{\alpha}}{\alpha} - \frac{\theta}{3} - \xi\right) Q + \left(-\frac{\theta}{3} + \frac{\alpha}{2}z\right) P + \frac{1}{2}\alpha g,$$
$$\dot{P} = \left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta\right) P, \tag{371}$$

associated with the Hamiltonian

$$\mathcal{H}(Q,P) = \frac{1}{2}\left(-\frac{\theta}{3} + \frac{\alpha}{2}z\right)P^2 - \left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta\right) Q P + \frac{\alpha}{2} g P. \tag{372}$$

The third case is equivalent to the situation described by (354a–354e) with new variables and can be written in the form

$$\dot{Q} = \left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta - \xi + \frac{\alpha}{2}y\right) Q + \left(-\frac{\theta}{3} + \frac{\alpha}{2}y\right) P + \frac{\alpha}{2} g,$$
$$\dot{P} = -\frac{\alpha}{2} y Q + \left(\frac{\dot{\alpha}}{\alpha} - \frac{\theta}{3} - \frac{\alpha}{2}y\right) P - \frac{\alpha}{2} g. \tag{373}$$

The Hamiltonian associated with this case is

$$\mathcal{H}(Q,P) = \frac{\alpha}{4} y Q^2 + \frac{1}{2}\left(-\frac{\theta}{3} + \frac{\alpha}{2}y\right)P^2 - \left(\frac{\dot{\alpha}}{\alpha} - \frac{\theta}{3} - \frac{\alpha}{2}y\right) Q P + \frac{\alpha}{2} g (Q + P). \tag{374}$$

3.5.4 The Specific Solutions

We proceed to study the three particular cases presented in Section 3.5.1, where a degree of freedom was lost to eliminate the acceleration Ψ.

In the first case (isotropic or shear-free model), we have $\Sigma = 0$ and, with (356) in the system (354a–354e)–(355a–355c), we obtain the following results

$$\begin{aligned} H(t) &= C_1 a^{-2}(t), \\ E(t) &= -\frac{2C_1}{3}\theta a^{-2}(t), \\ \Omega(t) &= -2 C_1 a^{-2}(t), \\ \Psi(t) &= -\frac{4C_1}{3}\theta a^{-2}(t), \\ q(t) &= -2C_1 a^{-2}(t)\left[(m + 2\epsilon) a^{-2}(t) + 2(\rho + p)\right], \end{aligned} \tag{375}$$

where C_1 is an integration constant.

A nonzero heat flux is needed for a shear-free linear perturbation, since zero shear is a characteristic condition for no perturbation in the perfect-fluid case (cf. Goode, [59]).

The second case (irrotational model, $\Omega = 0$) gives, upon substitution of (357) in (354a–354e)–(355a–355c), the following results:

$$\begin{aligned} \Sigma(t) &= C_2 a^{-2}(t) e^{-\xi t}, \\ H(t) &= -\frac{C_2}{2} a^{-2}(t) e^{-\xi t}, \\ E(t) &= C_2\left(\frac{\theta}{3} + \frac{\xi}{2}\right) a^{-2}(t) e^{-\xi t}, \\ q(t) &= C_2(m + 2\epsilon) a^{-4}(t) e^{-\xi t}, \end{aligned} \tag{376}$$

where C_2 is another integration constant.

In this case, the fluid must be nonperfect to allow a linear perturbation with zero vorticity.

Finally, for the third case (Stokesian fluid), with $q = 0$, $p = \lambda\rho$ and (358) being valid, the system (354a–354e)–(355a–355c) yields the reduced dynamics

$$\dot{\Sigma} = -\left[\frac{2}{3}\theta + \xi\left(1 + \frac{1}{2a^2}\frac{(m + 2\epsilon)}{(\rho + p)}\right)\right]\Sigma - (1 - 3\lambda)\frac{\theta}{3}\Omega,$$
$$\dot{\Omega} = \frac{1}{2a^2}\frac{(m + 2\epsilon)}{(\rho + p)}\xi\Sigma - (1 + 3\lambda)\frac{\theta}{3}\Omega. \tag{377}$$

We again seek a Hamiltonian description with variables (Q, P), using the transformation in (363). Differentiating these expressions, we find that (377) can be rewritten as

$$\begin{aligned}
\dot{Q} &= \left\{\dot{\alpha} - \left[\frac{2}{3}\theta + \xi\left(1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right)\right]\alpha \right.\\
&\quad \left. + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\xi\beta\right\}\frac{1}{\Delta}(\delta Q - \beta P)\\
&\quad + \left\{\dot{\beta} - [\alpha(1-3\lambda) + \beta(1+3\lambda)]\frac{\theta}{3}\right\}\frac{1}{\Delta}\\
&\quad \times(-\gamma Q + \alpha P),\\
\dot{P} &= \left\{\dot{\gamma} - \left[\frac{2}{3}\theta + \xi\left(1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right)\right]\gamma \right.\\
&\quad \left. + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\xi\delta\right\}\frac{1}{\Delta}(\delta Q - \beta P)\\
&\quad + \left\{\dot{\delta} - [\gamma(1-3\lambda) + \delta(1+3\lambda)]\frac{\theta}{3}\right\}\frac{1}{\Delta}\\
&\quad \times(-\gamma Q + \alpha P).
\end{aligned} \tag{378}$$

From (378), we read the Hamiltonian constraint

$$\begin{aligned}
\Phi &\equiv \Delta\left(\frac{\partial\dot{Q}}{\partial Q} + \frac{\partial\dot{P}}{\partial P}\right)\\
&= \dot{\Delta} - \left[\frac{\theta}{3} + \xi\left(1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right)\right]\Delta + \lambda\theta\Delta = 0,
\end{aligned} \tag{379}$$

whose solution is given by the expression

$$\begin{aligned}
\Delta(t) &= a^{(1-3\lambda)}(t)\exp\xi\int_{(H_0^{-1}+c_0)}^{t}\\
&\quad \times\left\{1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(1+\lambda)\rho(t')}\right\}dt',
\end{aligned} \tag{380}$$

where c_0 is again a positive integration constant.

We now set the Hamiltonian variables (Q, P) as given by (363) with

$$\alpha = \delta = \Delta^{1/2},$$
$$\beta = \gamma = 0,$$

where Δ is given by (380).

We finally obtain the dynamics

$$\begin{aligned}
\dot{Q} &= \left\{\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta - \xi\left[1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right]\right\}Q\\
&\quad - \left\{(1-3\lambda)\frac{\theta}{3}\right\}P,\\
\dot{P} &= \left[\frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\xi\right]Q + \left[\frac{\dot{\alpha}}{\alpha} - (1-3\lambda)\frac{\theta}{3}\right]P,
\end{aligned} \tag{381}$$

under the constraint of vanishing heat flux

$$Q = -\left[1 + \frac{2a^2}{(m+2\epsilon)}(\rho+p)\right]P. \tag{382}$$

The associated Hamiltonian is then given by the equation

$$\begin{aligned}
\mathcal{H}(Q, P) &= -\frac{1}{2}\left[\frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\xi \right.\\
&\quad \left. + \frac{2}{3}\frac{(1+3\lambda)\theta}{1 + \frac{2a^2}{(m+2\epsilon)}(\rho+p)}\right]Q^2 +\\
&\quad -(1-3\lambda)\frac{\theta}{3}P^2 - \left\{(1+\lambda)\frac{\theta}{2}\right.\\
&\quad \left. + \frac{\xi}{2}\left[1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right]\right\}QP..
\end{aligned} \tag{383}$$

As an example, the equations of motion (381) can be explicitly integrated by taking (382) into account. Thus, the system evolution follows the equation

$$\dot{P} = -\left\{(1+9\lambda)\frac{\theta}{6} + \frac{\xi}{2}\left[1 + \frac{1}{2a^2}\frac{(m+2\epsilon)}{(\rho+p)}\right]\right\}P, \tag{384}$$

which can be readily integrated and we finally find that

$$\begin{aligned}
Q &= -\left[1 + \frac{2a^2}{(m+2\epsilon)}(\rho+p)\right]a^{-\frac{(1+9\lambda)}{2}}\\
&\quad \exp\left\{-\frac{\xi}{2}\int_{(H_0^{-1}+c_0)}^{t}\left[1 + \frac{1}{2a^2(t')}\frac{(m+2\epsilon)}{(1+\lambda)\rho(t')}\right]dt'\right\},\\
P &= a^{-\frac{(1+9\lambda)}{2}}\exp\left\{-\frac{\xi}{2}\int_{(H_0^{-1}+c_0)}^{t}\left[1 + \frac{1}{2a^2(t')}\right.\right.\\
&\quad \left.\left. \times\frac{(m+2\epsilon)}{(1+\lambda)\rho(t')}\right]dt'\right\},
\end{aligned} \tag{385}$$

Back to the physically relevant variables, we find, in particular, that

$$\Omega(t) = a^{-(1+3\lambda)} \exp\left\{-\xi \int_{\left(H_0^{-1}+c_0\right)}^{t} \left[1 + \frac{1}{2a^2} \times \frac{(m+2\epsilon)}{(1+\lambda)\,\rho(t')}\right] dt'\right\}. \quad (386)$$

The perturbation in the vorticity appears to diverge and hence to break down our fundamental approach of linear treatment, for perturbation wavelengths such that

$$m < -2\epsilon - 2\,(1+\lambda)\,a^2\,\rho. \quad (387)$$

However, (342) shows that $m > -2\epsilon$, always, so that $\Omega \to 0$, a result that we could have expected from the angular-momentum conservation law.

We therefore find the minimal set of observables for the vectorial mode:

$$\mathcal{M}_A^{vector} = \{\Sigma, \Omega, q, \Psi\}.$$

The system is not closed, however, since the variable Ψ cannot be written in terms of the other ones. In order to solve the system, we therefore have to eliminate one of the variables and lose a degree of freedom.

3.6 Friedman Universe: Tensorial Perturbation

Here, we will proceed as in Section 3.5.1 to derive an ordinary differential system that describes tensorial perturbations in terms of *good* variables.

The tensorial basis $\hat{U}_{\alpha\beta}(x)$ is defined by the relations

$$\begin{aligned}
\dot{\hat{U}}_{\alpha\beta} &= 0,\\
h^{\mu\nu}\,\hat{U}_{\mu\nu} &= 0,\\
\hat{\nabla}^{\mu}\,\hat{U}_{\mu} &= 0,\\
\hat{U}_{\alpha\beta} &= \hat{U}_{\beta\alpha},\\
\hat{\nabla}^2\,\hat{U}_{\alpha\beta} &= \frac{m}{A^2}\,\hat{U}_{\alpha\beta},
\end{aligned} \quad (388)$$

where the new eigenvalue m has the following spectrum

$$m = \begin{cases} q^2+3, & 0 < q < \infty, \quad \epsilon = +1 \text{ (open)},\\ q, & 0 < q < \infty, \quad \epsilon = \ \ 0 \text{ (plane)},\\ n^2-3, & n = 3, 4, \ldots, \quad \epsilon = -1 \text{ (closed)}. \end{cases} \quad (389)$$

Using the tensor basis, we can define the dual tensor

$$\hat{U}^*_{\mu\nu} \equiv \frac{1}{2}\,h^{\alpha}_{(\mu}\,h^{\beta}_{\nu)}\,\eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\alpha}. \quad (390)$$

We employ the following tensorial relations to obtain the dynamical equations system:

$$\begin{aligned}
\dot{\hat{U}}^*_{\alpha\beta} &= -\frac{1}{3}\,\theta\,\hat{U}^*_{\alpha\beta},\\
\hat{U}^{**}_{\alpha\beta} &= \left(\frac{m}{a^2} + \rho - \frac{1}{3}\,\theta^2\right)\hat{U}_{\alpha\beta} = \frac{1}{a^2}\,(m-3\epsilon)\,\hat{U}_{\alpha\beta},
\end{aligned} \quad (391)$$

which involves the energy density ρ and the expansion coefficient θ.

We now expand the *good* perturbed quantity on the above basis to find the expression

$$\begin{aligned}
\delta\sigma_{\alpha\beta} &= \Sigma(t)\,\hat{U}_{\alpha\beta}\\
\delta H_{\alpha\beta} &= H(t)\,\hat{U}^*_{\alpha\beta}\\
\delta E_{\alpha\beta} &= E(t)\,\hat{U}_{\alpha\beta}\\
\delta\pi_{\alpha\beta} &= \pi(t)\,\hat{U}_{\alpha\beta},
\end{aligned} \quad (392)$$

where the time-dependent functions Σ, E, H and π are unrelated to the vector components of the previous section.

3.6.1 Dynamics

Under the properties (388)–(391) and again making use of (349), the quasi-Maxwellian equations are written in the form

$$\dot{E} - \frac{\xi}{2}\,\dot{\Sigma} + \theta\,E - \frac{1}{2}\left[\frac{\theta}{3}\,\xi - (\rho+p)\right]\Sigma + \frac{1}{a^2}\,(m-3\epsilon)\,H = 0, \quad (393a)$$

$$\dot{H} + \frac{2}{3}\,\theta\,H + E + \frac{\xi}{2}\,\Sigma = 0, \quad (393b)$$

$$\dot{\Sigma} + \left(\frac{2}{3}\,\theta + \frac{\xi}{2}\right)\Sigma + E = 0, \quad (393c)$$

constrained to

$$\eta \equiv H - \Sigma = 0. \quad (394)$$

We also know that Φ is dynamically preserved as follows:

$$\dot{\eta} = \chi_2 - \chi_3 - \frac{2}{3}\,\theta\,\eta, \quad (395)$$

where χ_2 and χ_3 are (393b) and (393c), respectively.

From this we are, therefore, authorized to insert it into dynamics, a procedure that leads to the unconstrained coupled differential system

$$\dot{E} + \left(\theta + \frac{\xi}{2}\right)E + \left\{\frac{1}{2}\left[\xi\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + (\rho+p)\right] - \frac{1}{a^2}\,(m-3\epsilon)\right\} H = 0, \quad (396a)$$

$$\dot{H} + \left(\frac{2}{3}\,\theta + \frac{\xi}{2}\right) H + E = 0. \quad (396b)$$

The coefficient H in (396a) in the de Sitter background yields a positive[25] constant leading term, for times such that $(1/a^2) \simeq 0$. This feature will be important in Section 3.6.2.

We also stress that (396a)–(396b) have nontrivial solution unless both (E, H) are assumed to be nonzero. That is, both variables are essential in describing tensor perturbations—it should be remembered that these variables constitute the electric and magnetic parts of Weyl tensor.

3.6.2 Hamiltonian Treatment of the Tensorial Solution

The basic system given by (396a)–(396b) can be described in the Hamiltonian language, which provides a more elegant interpretation of the dynamical role of our variables. The link between it and perturbation theory has worth on its own. We thus introduce new variables

$$\begin{pmatrix} Q \\ P \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}, \tag{397}$$

where we suppose

$$\Delta \equiv \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma \neq 0,$$

which is proven a posteriori to be actually correct. Therefore, we can use the set (Q, P) for (E, H) in order to characterize the tensorial perturbations. Inserting definitions (397) into (396a)–(396b), we eventually get the result

$$\begin{aligned} \dot{Q} =& \left\{\dot{\alpha} - \alpha\left(\theta + \frac{\xi}{2}\right) - \beta\right\} E + \left\{\dot{\beta} - \beta\left(\frac{2}{3}\theta + \frac{\xi}{2}\right)\right. \\ & -\alpha\left(\frac{1}{2}\left[\xi\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + (\rho + p)\right]\right. \\ & \left.\left. - \frac{1}{a^2}(m - 3\epsilon)\right)\right\} H, \\ \dot{P} =& \left\{\dot{\gamma} - \gamma\left(\theta + \frac{\xi}{2}\right) - \delta\right\} E + \left\{\dot{\delta} - \delta\left(\frac{2}{3}\theta + \frac{\xi}{2}\right)\right. \\ & -\gamma\left(\frac{1}{2}\left[\xi\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + (\rho + p)\right]\right. \\ & \left.\left. - \frac{1}{a^2}(m - 3\epsilon)\right)\right\} H. \end{aligned} \tag{398}$$

We also need to show that our variables are, in fact, canonically conjugated to each other, as suggested by the notation. That is, we again make use of the Hamiltonian constraint

$$\Phi \equiv \Delta\left(\frac{\partial\dot{Q}}{\partial Q} + \frac{\partial\dot{P}}{\partial P}\right) = \dot{\Delta} - \left(\frac{5}{3}\theta + \xi\right)\Delta = 0. \tag{399}$$

A particular solution of (399) is

$$\Delta(t) = a^5(t)\, e^{\xi t}, \tag{400}$$

and we then set

$$\begin{aligned} \alpha &= \Delta^{\omega}, \\ \delta &= \Delta^{(1-\omega)}, \\ \beta &= \gamma = 0, \end{aligned} \tag{401}$$

where ω is an arbitrary constant.

With the choice (401) and using solution (400), system (398) becomes

$$\begin{aligned} \dot{P} =& -\left[\left(\frac{5}{3}\omega - 1\right)\theta + \left(\omega - \frac{1}{2}\right)\xi\right] P - \Delta^{(1-2\omega)} Q \\ \dot{Q} =& -\left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) - \frac{1}{2}(\rho + p)\right. \\ & \left. - \frac{1}{2a^2}(m - 3\epsilon)\right] \Delta^{(2\omega-1)} P \\ & + \left[\omega\left(\frac{5}{3}\theta + \xi\right) - \left(\theta + \frac{\xi}{2}\right)\right] Q. \end{aligned} \tag{402}$$

From this, we directly read the Hamiltonian

$$\begin{aligned} \mathcal{H}(Q, P) =& -\frac{1}{2}\Delta^{(2\omega-1)}\left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p)\right. \\ & \left. - \frac{1}{a^2}(m - 3\epsilon)\right] P^2 \\ & + \frac{1}{2}\Delta(1 - 2\omega)\, Q^2 + \left[\left(\frac{5}{3}\omega - 1\right)\theta\right. \\ & \left. + \left(\omega - \frac{1}{2}\right)\xi\right] PQ. \end{aligned} \tag{403}$$

This result shows that the de Sitter ($\theta = const.$) geometry admits a tensor perturbation Hamiltonian of a typical harmonic oscillator with imaginary mass, which evidences instability. This is obtained by setting the arbitrary constant parameter

$$\omega = \frac{3}{2}\frac{(2\theta + \xi)}{(5\theta + 3\xi)},$$

in the Hamiltonian (403).

We thus recover the well-known result of the instability of the de Sitter solution. The above result also shows, however, that the same remark applies to arbitrary Friedman-like backgrounds with no tensorial perturbation in anisotropic pressure tensor, $\xi = 0$. In such cases, we set $\omega = 3/5$ to find that

$$\begin{aligned} \mathcal{H}(Q, P)\Big|_{\xi=0} =& -\frac{1}{2}\Delta^{1/5}\left[(\rho + p) - \frac{1}{a^2}(m - 3\epsilon)\right] P^2 \\ & + \frac{1}{2}\Delta^{-1/5} Q^2, \end{aligned} \tag{404}$$

where (Q, P) are given by the equality

$$\begin{aligned} Q &= a^3(t)\, e^{\frac{3}{5}\xi t} E \\ P &= a^2(t)\, e^{\frac{2}{5}\xi t} H. \end{aligned} \tag{405}$$

[25] Astronomical observations show that the Hubble constant, here translated to θ, is positive, even if there is no universal agreement on its magnitude. Thermodynamical reasoning ensures the nonnegativeness of the parameter ξ.

To summarize, we have found that there is a complete set of *good* perturbed variables for tensorial modes:

$$\mathcal{M}_A^{tensor} = \{E, H\}.$$

In this case, the system is closed and completely independent of the other modes, due to the linearity of the harmonic basis.

We have also obtained the Hamiltonian formulation for all modes, according to the previous sections and we can address the possibility to canonically quantize the cosmological perturbations of FLRW universes. The next section will discuss this analysis in more detail.

3.7 Friedman Universe: Quantum Treatment of the Perturbations

In Sections 3.4, 3.5 and 3.6, we have shown how to treat, in a completely gauge-invariant way, the evolution of the perturbations of FLRW universes. Besides, we have shown that it is possible to select a minimal set of observable quantities to analyze the perturbations of the FLRW universe. We have also shown that the complete dynamical system of the perturbed geometry is described only in terms of two quantities: E and Σ (respectively, the electric part of the conformal Weyl tensor and the shear), for scalar perturbations: E and H, (where the last quantity is the associated magnetic part of Weyl conformal tensor) for tensorial perturbations. For vectorial perturbations, in a more general case, this minimal set should be expanded to include E, Σ, H and the vorticity Ω. Now, we have completed the classical treatment of these gauge-independent perturbations; it is rather natural then to go beyond the classical theory. Indeed, the purpose of this section is to treat the perturbations in the quantum framework.

This amounts to using a semiclassical description in which the background geometry is taken in the classical framework and considering the perturbations as quantum variables. There are many ways to perform this task. Here, we will follow a very natural way that consists of applying the method of the auxiliary Hamiltonian, which was introduced before. The problem can be stated in the following way: using the quasi-Maxwellian formulation of Einstein's General Relativity, we find out that the complete dynamical system reduces to the form

$$\begin{bmatrix} \dot{M}_1 \\ \dot{M}_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \tag{406}$$

where $\mathcal{M}$ is a 2×2 matrix containing information that characterizes the background geometry and M_j are the "good" observables that describe the perturbations.[26] The dot ($\dot{}$) denotes a temporal derivative.

For the FLRW background, this set constitutes a nonautonomous dynamical system. Direct inspection of the matrix $\mathcal{M}$ shows that it is not trace-free; thus, this system has no Hamiltonian. Nevertheless, we have exhibited a method that allowed us to obtain an auxiliary Hamiltonian $\mathcal{H}$ for this system. As we will see later, the linear relation between the associated canonically conjugated variables (Q, P) and the original physical quantities (E, Σ, H, or Ω) is not unique. This is not a drawback of this approach, but merely a consequence of the fact that the set of possible pairs (Q, P) is related in turn by canonical transformations.

The existence of this Hamiltonian leads us to consider the possibility of employing the canonical method to arrive at the quantum version of the perturbed set. The quantum study of these perturbations in FLRW was done by Lifshitz [89], Hawking [65] and Novello [118]. However, all these previous works deal with variables which either are gauge-dependent or follow the general scheme introduced by Bardeen [9] and subsequent papers (cf. Ellis [50]), which has a difficult physical interpretation. This makes the analysis more complex.

Our method, in which the gauge problem is inexistent, seems to be really the best way to make the transition to the quantum version. Alternative methods, the minisuperspaces approach (e.g., Ryan [139]), for instance, suffer not only from needing to fix a gauge, but also from the fact that the order in which this choice is made (before or after quantization) leads to different theories. Even the schemes that consider gauge-independent variables (as initially defined by Bardeen) have, as a main problem, the absence of evident physical interpretation for the variables, which makes the physical comprehension of the results quite complex. Since the gauge-independent variables in our method are observables and completely equivalent to Bardeen's, the advantages of quantizing the dynamical systems obtained in our procedure are evident.

In order to perform the quantization of our system, we will make use of squeezed states of quantum optics, which was first employed in the framework of Cosmology by Grishchuk [63], Schumaker [140] and Bialynicha-Birula [16], in procedures that suffer from the same aforementioned difficulties. The advantages of including the method of gauge-independent variables in this approach are therefore obvious.

All the definitions and notations employed in obtaining the results for gauge-invariant, observable perturbations in the FLRW background (scalar, vectorial and tensorial) are

[26] Here, we are considering only the cases in which the minimal closed set of observables contains only two variables. In the more general vectorial case, $\mathcal{M}$ should be a 4×4 matrix, as stated above (see Section 3.7.3 for more details).

equally valid here. We therefore write the FLRW geometry in the standard Gaussian coordinate system.

For a comoving observer (one with $V^{\alpha} = \delta^{\alpha}_{0}$), we let ρ denote the energy density, p denote the isotropic pressure and θ denote the expansion. The constraint relation

$$-\frac{\epsilon}{a^2} - \frac{1}{3}\,\rho + \left(\frac{\theta}{3}\right)^2 = 0 \tag{407}$$

holds, along with the following auxiliary relation,

$$\rho = \rho_0\, a^{-3\,(1+\lambda)}, \tag{408}$$

which comes from the Raychaudhuri equation for a fluid with the usual linear state equation $p = \lambda\,\rho$. The parameter ρ_0 denotes the energy density for $a(t_0) = 1$.

We will use ξ to denote the viscosity. As before, we will consider it in the limit of small relaxation times for the adiabatic approximation of the thermodynamic equation (see Novello [122]) and, given this choice and thermodynamic considerations, will take it as a negative constant.

We will choose the geometrical units system, $\hbar \equiv k \equiv c = 1$. The constant m will denote the wave number associated to the perturbations in the FLRW background and the arbitrary integration constants κ and b will be employed throughout the section, for all three perturbation types. Additionally, we will denote by calligraphic letters the matrices (such as the Hamiltonian matrix $\mathcal{H}$) and by capital letters their linear counterparts (for example, the Hamiltonian H). When following the standard quantization procedure, in an effort to keep the notation simple, we will make no explicit indication (such as turning the Hamiltonian H into the Hamiltonian operator $\hat{H}$), except for the creation $(\hat{a}^{\dagger})$ and annihilation $(\hat{a})$ operators, which will distinguished from the the scale factor $a(t)$ by the " $(\widehat{\ })$ " symbol.

3.7.1 Auxiliary Hamiltonian

Consider the linear two-dimensional dynamical system for the variables M_1 and M_2, which gives the dynamics of evolution in the FLRW universe background for the minimal closed set of these gauge-independent linear perturbation quantities, given by the equality

$$\begin{bmatrix} \dot{M}_1 \\ \dot{M}_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \tag{409}$$

where $\mathcal{M}$ is a 2×2 matrix that may depend on time through the known background quantities and M_j are the observables forming the minimal closed set that describes all the perturbations. As it has been pointed out, the variables (M_1, M_2) are not canonically conjugated.

We thus define a new set of variables $(Q,\, P)$ as follows:

$$\begin{bmatrix} Q \\ P \end{bmatrix} = S \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \tag{410}$$

where α, β, γ and δ are functions of time and S, the transformation matrix, has an inverse and a determinant given by the expression

$$\Delta \equiv det(S) = \alpha\,\delta - \beta\,\gamma \neq 0, \tag{411}$$

As a consequence, the variables $(Q,\, P)$ satisfy the following dynamics:

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \mathcal{H} \begin{bmatrix} Q \\ P \end{bmatrix},$$

where $\mathcal{H}$ is a 2×2 matrix depending on time through $\mathcal{M}$ and the transformation matrix S.

From (409) and (410), it follows that

$$\mathcal{H} = S\,\mathcal{M}\,S^{-1} + \dot{S}\,S^{-1}.$$

If we require $(Q,\, P)$ to be canonical variables, then the matrix $\mathcal{H}$ must be traceless:

$$Tr\,\mathcal{H} = 0.$$

From the above equation, we can see that

$$Tr\,\mathcal{M} + \frac{\dot{\Delta}}{\Delta} = 0, \tag{412}$$

which can be easily integrated.

Thus, we have a set $(Q,\, P)$ of canonical variables, with an associated Hamiltonian H that is linearly related to $\mathcal{H}$. The above condition ensures that the set $(\alpha,\, \beta,\, \gamma,\, \delta)$ has only three independent quantities. These degrees of freedom are fixed by the canonical transformations, as it will be discussed later.

The most general quadratic Hamiltonian for our system can be written as

$$H = \frac{h_1}{2}\,Q^2 + \frac{h_2}{2}\,P^2 + 2\,h_3\,P\,Q, \tag{413}$$

the equivalent matrix form of which is

$$\mathcal{H} = \begin{bmatrix} 2\,h_3 & h_2 \\ -h_1 & -2\,h_3 \end{bmatrix}, \tag{414}$$

where h_i's are functions of α, β, γ and δ, as well as of quantities in the FLRW background.

Equation (414) allows us to decompose $\boldsymbol{\mathcal{H}}$ as $\boldsymbol{\mathcal{H}} = \vec{\mu}\,.\,\vec{\sigma}$, where $\vec{\mu}$ has the following components:

$$\left(\frac{(h_2 - h_1)}{2},\, \frac{i}{2}(h_1 + h_2),\, 2h_3\right).$$

The h_k $(k = 1, 2, 3)$ depend indirectly on the parameter t, through the known quantities of the FLRW background. The vector $\vec{\sigma}$ is built with the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Expressed in terms of background quantities, the h_k vary according to the perturbation type. They will be presented in future sections. For now, we have that

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} \Sigma \\ E \end{bmatrix}, \begin{bmatrix} \Sigma \\ \Omega \end{bmatrix}, \begin{bmatrix} E \\ H \end{bmatrix},$$

for scalar, vectorial and tensorial perturbations, respectively.[27]

Let us make a canonical transformation by changing the variables Q and P into $\tilde{Q}$ and $\tilde{P}$:

$$\begin{bmatrix} \tilde{Q} \\ \tilde{P} \end{bmatrix} = J \begin{bmatrix} Q \\ P \end{bmatrix}, \tag{415}$$

We then obtain a new Hamiltonian for the transformed system as a function of the previous one and of the transformation matrix J, that is,

$$\tilde{\mathcal{H}} = J\mathcal{H}J^{-1} + \dot{J}J^{-1}. \tag{416}$$

To guarantee that the system will still be described by a Hamiltonian, we require $\tilde{\mathcal{H}}$ to be traceless

$$Tr\left(\dot{J}J^{-1}\right) = 0,$$

that is,

$$det\ J = 1. \tag{417}$$

This is but the well known fact that quadratic Hamiltonians constitute the equivalence class of the harmonic oscillator. The group $SL(2,\mathbb{R})$ describes the canonical transformations on a plane.

3.7.2 The Scalar Case

The Auxiliary Hamiltonian In this section, we will present the results of the auxiliary-Hamiltonian method for the scalar perturbations of the FLRW background (cf. Novello [122, 123]). The resulting dynamical system for scalar perturbations in the general case (with nonzero viscosity ξ) is given by the equalities

$$\dot{\Sigma} = \xi\left[\frac{2m}{(1+\lambda)\rho a^2}\left(1+\frac{3\epsilon}{m}\right)\left(\frac{\lambda}{2}+\frac{1}{3}\right)-\frac{1}{2}\right]\Sigma - \left[1+\frac{2m\lambda}{(1+\lambda)\rho a^2}\left(1+\frac{3\epsilon}{m}\right)\right]E, \tag{418}$$

and

$$\dot{E} = \frac{1}{2}\left[\frac{2m}{(1+\lambda)\rho a^2}\left(1+\frac{3\epsilon}{m}\right)\left(\frac{\lambda}{2}+\frac{1}{3}\right)\xi^2 - \frac{\xi^2}{2} - (1+\lambda)\,\rho - \frac{\theta}{3}\,\xi\right]\Sigma - \left[\frac{m\lambda}{(1+\lambda)\rho a^2}\left(1+\frac{3\epsilon}{m}\right)\xi + \frac{\theta}{3} + \frac{\xi}{2}\right]E. \tag{419}$$

The components of the Hamiltonian matrix (414), h_k, are then written as follows:

$$\begin{aligned} h_1 &= \frac{1}{\Delta}\Bigg\{-\dot{\gamma}\,\delta + \gamma\,\dot{\delta} + \delta^2\,\frac{(1+\lambda)}{2}\,\rho - \gamma^2\,(1+\lambda\,L) - \gamma\,\delta\,\frac{\theta}{3} + \\ &- \frac{\xi}{2}\left[2\,\gamma\,\delta\,L\left(\lambda+\frac{1}{3}\right) - \delta^2\left(\frac{\xi}{2}\,(1-\lambda\,L) + \frac{(\theta-\xi\,L)}{3}\right)\right]\Bigg\}, \\ h_2 &= \frac{1}{\Delta}\Bigg\{\alpha\,\dot{\beta} - \dot{\alpha}\,\beta - \alpha^2\,(1+\lambda\,L) - \alpha\,\beta\,\frac{\theta}{3} + \beta^2\,\frac{(1+\lambda)}{2}\,\rho + \\ &- \frac{\xi}{2}\left[2\,\alpha\,\beta\,L\left(\lambda+\frac{1}{3}\right) - \beta^2\left(\frac{\xi}{2}\,(1-\lambda\,L) + \frac{(\theta-\xi\,L)}{3}\right)\right]\Bigg\} \\ h_3 &= \frac{1}{2\Delta}\Bigg\{-\alpha\,\dot{\delta} + \beta\,\dot{\gamma} - \beta\,\delta\,\frac{(1+\lambda)}{2}\,\rho + \alpha\,\gamma\,(1+\lambda\,L) + \alpha\,\delta\,\frac{\theta}{3} + \\ &- \frac{\xi}{2}\left[\alpha\,\delta\,(1+\lambda\,L) + \beta\,\gamma\left(\frac{2L}{3} - (1-\lambda\,L)\right) + \right. \\ &- \left.\beta\,\delta\left(\frac{\xi}{2}\,(1-\lambda\,L) + \frac{(\theta-\xi\,L)}{3}\right)\right]\Bigg\}. \end{aligned} \tag{420}$$

The auxiliary quantity L on the right-hand sides of (421) is defined as follows:

$$L \equiv \frac{2m}{(1+\lambda)\rho a^2}\left(1+\frac{3\epsilon}{m}\right).$$

Given (421), the condition (412) for the existence of the Hamiltonian now reads:

$$\frac{\dot{\Delta}}{\Delta} - \frac{\theta}{3} - \xi\left(1-\frac{L}{3}\right) = 0. \tag{421}$$

We will tackle the simpler case of zero viscosity for scalar perturbations in the FLRW background, which then yields

$$\Delta(t) = \kappa\, a(t), \tag{422}$$

where κ is an integration constant.

Canonical Quantization The problem to be analyzed at this point is but a single harmonic oscillator problem with a time-dependent quadratic interaction. This problem appears in many different contexts, e.g., the equation for quantum test fields in homogeneous and isotropic expanding/contracting universes, quantum optics, etc. There are many ways to face this problem; here, we will follow the standard procedure of quantum optics. The creation and

[27] In the specific case of a Stokesian fluid, the observables for vectorial perturbations yield a reduced dynamical system (for more information on that issue, see Novello [123]).

annihilation operators a and $a^\dagger$ are defined in the standard way as

$$\hat{a} = \frac{1}{\sqrt{2}}(Q + iP),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(Q - iP),$$
$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{423}$$

Using (413), (421) and (423), the Hamiltonian then becomes

$$H = H_0 + H_{int}, \tag{424}$$

with

$$H_0 \equiv \omega(t)(1 + 2N),$$
$$\omega(t) \equiv \frac{1}{4}(h_1 + h_2), \tag{425}$$

where N is the number of particles (photons): $N \equiv \hat{a}^\dagger \hat{a}$. The self-interaction Hamiltonian is given by the expression

$$H_{int} \equiv \eta(t)\hat{a}^2 + \eta^\star(t)(\hat{a}^\dagger)^2, \tag{426}$$

where

$$\eta(t) \equiv \frac{1}{4}(h_1 - h_2) - ih_3, \tag{427}$$

and $\eta^\star$ is the complex conjugate of η.

Schrödinger equation is then easily written for the operator $\mathcal{H}$ as

$$i\,\frac{\partial \psi(\vec{x}, t)}{\partial t} = H\,\psi(\vec{x}, t),$$

with the wave function $\psi(\vec{x}, t)$ given by the equality $\psi(\vec{x}, t) = U(t, t_0)\,\psi(\vec{x}, t_0)$, where $U(t, t_0)$ is the evolution operator.

We will now proceed to solve the equation above by employing the quantum-optics formalism. This involves writing the time evolution operator as a product of the rotation and the single-mode squeeze operators, along with a phase factor:

$$U(t, t_0) = e^{i\phi}\, S_{(r,\varphi)}\, R_{(\Gamma)},$$

where the phase ϕ is time dependent and the rotation operator $R_{(\Gamma)}$ and the single-mode squeeze operator $S_{(r,\varphi)}$ are defined as

$$R_{(\Gamma)} \equiv \exp\left(-i\Gamma\,\hat{a}\,\hat{a}^\dagger\right),$$
$$S_{(r,\varphi)} \equiv \exp\left\{\frac{r}{2}\left[e^{-2i\varphi}\,\hat{a}^2 - e^{2i\varphi}\,(\hat{a}^\dagger)^2\right]\right\},$$

respectively, where Γ, r and φ depend of time through the known quantities in the FLRW background and are defined as the rotation angle, the squeeze factor and the squeeze angle, respectively. It should be remarked that all these quantities are real. For further details and explanations, the reader should see Novello et al. [124].

A direct albeit somewhat long calculation reduces the Schrödinger equation to the following first-order coupled differential system:

$$\phi(t) = \frac{1}{2}\,\theta(t),$$
$$\dot{\Gamma} = \frac{2\omega}{\cosh(2r)},$$
$$\dot{r} = \frac{1}{2}(h_1 - h_2)\,\sin(2\varphi) - 2\,h_3\,\cos(2\varphi),$$
$$r\,\dot{\varphi} = \frac{1}{4}(h_1 - h_2)\,\cos(2\varphi) + h_3\,\sin(2\varphi) - \omega(t)\,\tanh(2r), \tag{428}$$

where $\omega(t)$ is defined by (425).

We now proceed to solve the system (428). The last two equations being coupled, their integration is very much involved. However, they can be easily integrated if we first transform the Hamiltonian matrix $\boldsymbol{\mathcal{H}}$ by means of a canonical transformation matrix $\mathcal{J}$, given by (416) and (417), such that the transformed Hamiltonian acquire the form

$$\tilde{\omega} \equiv \frac{1}{4}\left(\tilde{h}_1 + \tilde{h}_2\right) = 0, \tag{429}$$

We will drop the $(\tilde{\ })$ symbol from now on to simplify the notation. This transformation can always be carried out, since it only amounts to appropriately choosing the original functions α, β, γ and δ. The first step taken is then to choose

$$\mathcal{S} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \kappa\,a/\alpha \end{pmatrix}, \tag{430}$$

where κ is an integration constant that comes from the condition of existence of $\mathcal{H}$, (421).

The condition (429) may then be written as

$$\frac{(1+\lambda)}{2}\,\rho\left(\frac{\kappa a}{\alpha}\right)^2 - (1 + \lambda L)\,\alpha^2 = 0,$$

which then gives

$$\alpha^4(t) = \frac{\kappa^2(1+\lambda)\rho_0}{2a^{(1+3\lambda)}} \left\{1 + \frac{2\lambda}{(1+\lambda)\rho_0} \times (m+3\epsilon)\, a^{(1+3\lambda)}\right\}^{-1}, \tag{431}$$

where ρ_0 is the matter density when $a(t_0) = 1$.

The right-hand side of (431) is always positive, since $m > -3\epsilon$ in all circumstances. We therefore have the following results:

$$\begin{aligned} h_1 &= \frac{\kappa^2(1+\lambda)\rho}{2a}\alpha^{-2} = -h_2, \\ h_3 &= \frac{1}{2}\frac{\dot\alpha}{\alpha}, \\ \eta(t) &\equiv \frac{1}{4}(h_1 - h_2) - i h_3 = \frac{\kappa^2(1+\lambda)\rho}{4a\alpha^2} - \frac{i}{2}\frac{\dot\alpha}{\alpha}. \end{aligned} \tag{432}$$

We now proceed to integrating the differential equations in r and φ, which yields:

$$r\, e^{-2i\varphi} = -2i \int \eta(t)\, dt.$$

To make the integration on the right-hand side simpler, we will choose the case of spatial curvature zero and $\lambda = 1/3$, to find[28]

$$\int \eta(t)\, dt = -\frac{\kappa}{2}\left\{\sqrt{\frac{m}{\rho_0 a^2} + \frac{2}{a^4}} + \frac{\sqrt{2}}{4}\frac{m}{\rho_0} \ln\left[\sqrt{\frac{2\rho_0}{ma^2}} + \sqrt{1 + \frac{2\rho_0}{ma^2}}\right]\right\} + \frac{i}{4}\left\{\ln a(t) + \frac{1}{2}\ln\left(a^2 + \frac{2\rho_0}{m}\right)\right\}. \tag{433}$$

With the result in (433), it is a simple matter to decouple the differential equations to find the expressions

$$\begin{aligned} r\,\sin(2\varphi) &= -\sqrt{2}\kappa\left\{\frac{1}{a}\sqrt{\frac{m}{2\rho_0}a^2 + 1} + \frac{m}{4\rho_0} \ln\left(\sqrt{\frac{2\rho_0}{ma^2}} + \sqrt{\frac{2\rho_0}{ma^2} + 1}\right)\right\} \\ r\cos(2\varphi) &= \frac{1}{4}\ln\left[a^4\left(1 + \frac{2\rho_0}{m}\right)\right]. \end{aligned} \tag{434}$$

Observables From the above construction, it follows that the observables of the theory are written in terms of the corresponding creation and annihilation operators in the same modes.

[28]To be compatible with most of the cosmological models, in this section, we assume that the quantum phase of the Universe is radiation dominated. Note that the calculations can be easily extended for the case in which a previous inflationary regime is present.

For the shear Σ and for the electric part of Weyl tensor E, we have

$$\begin{aligned} \Sigma &= \chi(t)\hat a + \chi^\star(t)\hat a^\dagger, \\ E &= \Psi(t)\hat a + \Psi^\star(t)\hat a^\dagger, \end{aligned} \tag{435}$$

where $\chi(t)$ and $\Psi(t)$ are defined as follows:

$$\begin{aligned} \chi(t) &\equiv \frac{1}{\sqrt{2\Delta}}(\delta + i\beta), \\ \Psi(t) &\equiv \frac{1}{\sqrt{2\Delta}}(-\gamma + i\alpha), \end{aligned} \tag{436}$$

with $\alpha,\ \beta,\ \gamma,\ \delta$ the same quantities defined by (410).

On the basis of the same solution, (430) and (431), we easily find that

$$\begin{aligned} \chi(t) &= \frac{1}{\sqrt{2}\alpha} = \chi^\star(t), \\ \Psi(t) &= i\,\frac{\alpha}{\sqrt{2}\kappa a(t)} = -\Psi^\star(t). \end{aligned} \tag{437}$$

The commutator is then easily calculated to give

$$[\Sigma, E] = -i\,\frac{1}{\kappa a(t)}, \tag{438}$$

if the choice $\hbar \equiv c \equiv k = 1$ holds.

The total noise of the observables Σ and E can be calculated as follows:[29]

$$\begin{aligned} \langle\psi||\Delta\Sigma|^2|\psi\rangle &= \cosh(2r)\,\langle 0||\Delta\Sigma|^2|0\rangle, \\ \langle\psi||\Delta E|^2|\psi\rangle &= \cosh(2r)\,\langle 0||\Delta E|^2|0\rangle, \end{aligned} \tag{439}$$

where $\langle\psi||\Delta X|^2|\psi\rangle$ is the total noise calculated at the time t, $\langle 0||\Delta X|^2|0\rangle$ is the total noise in the vacuum state and $r(t)$ is given by (434).

3.7.3 The Vectorial Case

In this case, we find that the resulting dynamical system is not closed: it depends on the choice of the perturbation in the acceleration Ψ. Three different reduced dynamics have been studied:

- *Stokesian fluid:* $q = 0$; $p = \lambda\rho$
- *Shear-free model*
- *Vorticity-free model*

The second and third models give very simple, directly integrable results. The Stokesian fluid model implies, for the perturbed acceleration Ψ, that $\Psi = 2\lambda\,\theta\,\Omega$ and we obtain a closed reduced dynamics for the observables Σ and Ω (the

[29]This quantity is defined as the mean-square uncertainty in the annihilation operator $\hat a$. The total noise of a Gaussian pure state is conserved even if the total number of photons is not and, it is therefore more useful to describe the quantum wave functions obtained from the Schrödinger equation. See Novello et al. [124] for more details on this.

shear and the vorticity, respectively). This is the case that will be quantized here.

Auxiliary Hamiltonian As for the scalar case, we here present the results of the auxiliary-Hamiltonian method.

The reduced closed dynamical system for the special case of Stokesian fluid with nonzero viscosity and zero heat flux q is:

$$\dot{\Sigma} = -\left\{\frac{2}{3}\theta + \xi\left[1 + \frac{(m+2\epsilon)}{2(1+\lambda)\rho a^2}\right]\right\}\Sigma \\ -(1-3\lambda)\frac{\theta}{3}\Omega,$$

$$\dot{\Omega} = \xi\frac{(m+2\epsilon)}{2(1+\lambda)\rho a^2}\Sigma - (1+3\lambda)\frac{\theta}{3}\Omega, \tag{440}$$

where we must have $q = 0$, $p = \lambda\rho$ and $\Psi = 2\lambda\,\theta\,\Omega$. The condition for the existence of a Hamiltonian then reads

$$\Delta(t) = \kappa\, a^{3(1+\lambda)}\, e^{\xi\, M(t)}, \tag{441}$$

where κ is again an integration constant and $M(t)$ is an auxiliary quantity, defined by the equality

$$M(t) = \int_t \left[1 + \frac{(m+2\epsilon)}{2(1+\lambda)\rho a^2}\right] dt.$$

We can now proceed to the quantization formalism described by (423)–(428). With the choice

$$\beta \equiv \gamma = 0,$$
$$\delta = \frac{\Delta(t)}{\alpha},$$
$$\alpha^4 = -\frac{3\xi}{2}\,\frac{(m+2\epsilon)}{(1+\lambda)(1-3\lambda)}\,\frac{\kappa\, a^{(7+9\lambda)}\, e^{2\xi M(t)}}{\rho_0\theta}, \tag{442}$$

we obtain the following Hamiltonian coefficients h_j, $(j = 1, 2, 3)$:

$$h_1(t) = -h_2(t) = -\sqrt{-\frac{\xi}{6}\,\frac{(m+2\epsilon)\theta}{\rho_0}\, a^{(1+3\lambda)}\left(\frac{1-3\lambda}{1+\lambda}\right)},$$
$$h_3(t) = \frac{1}{2}\left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta\right). \tag{443}$$

Since the viscosity ξ is negative, due to thermodynamic considerations and since $m > -2\epsilon$ in all cases (see Novello [123]), the function $\alpha(t)$ is real.

We are then able to decouple the differential system resulting from Schrödinger (428), to find that

$$r\,\cos(2\varphi) = \ln\left(b\,\frac{a^2}{\alpha}\right),$$
$$r\,\sin(2\varphi) = \frac{3}{\rho_0}\sqrt{\xi\,\frac{(3\lambda-1)(m+2\epsilon)}{6(1+\lambda)}}\, a^{(1+3\lambda)}, \tag{444}$$

with α given by (442) and b being an integration constant.

Observables The same method applied to the scalar observables can be employed here to the vectorial case, with the results

$$\Sigma = \frac{1}{\alpha}\, Q = \frac{1}{\sqrt{2}\,\alpha}\,(\hat{a} + \hat{a}^\dagger),$$
$$\Omega = \frac{\alpha\, e^{-\xi M(t)}}{\sqrt{2}\kappa a^{3(1+\lambda)}}\, P = -i\,\frac{\alpha\, e^{-\xi M(t)}}{\sqrt{2}\kappa a^{3(1+\lambda)}}\,(\hat{a} - \hat{a}^\dagger). \tag{445}$$

And in this case, the commutator between the perturbed variables is expressed by the following expression:

$$[\Sigma, \Omega] = i\,\frac{e^{-\xi M(t)}}{\kappa a^{3(1+\lambda)}}.$$

Finally, the relation between the total noises of Σ and Ω at the time t and their total noises in the vacuum state is given by the equations

$$\langle\Psi||\Delta\Sigma|^2|\Psi\rangle = \cosh(2r)\,\langle 0||\Delta\Sigma|^2|0\rangle,$$
$$\langle\Psi||\Delta\Omega|^2|\Psi\rangle = \cosh(2r)\,\langle 0||\Delta\Omega|^2|0\rangle,$$

where r is given by (444).

The Case $\xi = 0$ If we consider the special case of vectorial perturbations in a Stokesian fluid with zero viscosity ($\xi = 0$), we can follow the same steps detailed in previous subsections to find the Hamiltonian coefficients

$$h_1 = -h_2 = 0,$$
$$h_3 = \frac{1}{2}\left(\frac{\dot{\alpha}}{\alpha} - \frac{2}{3}\theta\right), \tag{446}$$

where we have the choices

$$\alpha \equiv \text{arbitrary function of time},$$
$$\beta \equiv \gamma = 0,$$
$$\delta = \frac{\kappa}{\alpha}\, a(t)^4,$$

and the condition $\lambda = 1/3$ must hold, so that $\omega(t)$ is zero and the system that arises from the Schrödinger (428) is easily decoupled to yield the results

$$r = \ln\left(b\,\frac{a^2(t)}{\alpha(t)}\right),$$
$$\varphi = 0, \qquad \text{or} \qquad \varphi = -\pi, \tag{447}$$

with $b \equiv$ const., again.

The condition on λ ensures that our model for vectorial perturbations in a Stokesian fluid with zero viscosity only applies to the radiation era.

The observables Σ and Ω are then written

$$\Sigma = \frac{1}{\alpha(t)}\, Q = \frac{1}{\sqrt{2}\alpha(t)}\,(\hat{a} + \hat{a}^\dagger),$$
$$\Omega = \frac{\alpha(t)}{\kappa a^4(t)}\, P = -i\,\frac{\alpha(t)}{\sqrt{2}\kappa a^4(t)}\,(\hat{a} - \hat{a}^\dagger), \tag{448}$$

if $\hbar \equiv k \equiv c = 1$.

The commutator between Σ and Ω will then be

$$[\Sigma, \Omega] = -i \, \frac{1}{\kappa a^4(t)}, \tag{449}$$

and the total noises of the observables will be related to their values in the vacuum state as follows:

$$\begin{aligned} \langle\Psi||\Delta\Sigma|^2|\Psi\rangle &= \cosh\left(2 \ln\left(b \, \frac{a^2(t)}{\alpha(t)}\right)\right) \langle 0||\Delta\Sigma|^2|0\rangle, \\ \langle\Psi||\Delta\Omega|^2|\Psi\rangle &= \cosh\left(2 \ln\left(b \, \frac{a^2(t)}{\alpha(t)}\right)\right) \langle 0||\Delta\Omega|^2|0\rangle. \end{aligned} \tag{450}$$

3.7.4 The Tensorial Case

In this case, we obtain a new dynamical closed system for the observables E and H (the electric and magnetic parts of Weyl tensor, respectively) as follows:

$$\begin{aligned} \dot{E} &= -\left(\theta + \frac{\xi}{2}\right) E - \left\{\frac{1}{2}\left[\xi\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + (\rho + p)\right] \right. \\ &\quad \left. -\frac{1}{a^2}(m - 3\epsilon)\right\} H, \\ \dot{H} &= -E - \left(\frac{2}{3}\theta + \frac{\xi}{2}\right) H. \end{aligned} \tag{451}$$

The transformation to the variables (Q, P) follows the procedure in the scalar and vectorial cases, (410)–(414). The condition for the existence of a Hamiltonian is then

$$\Delta(t) = \kappa \, a^5(t) \, e^{\xi \, t}, \tag{452}$$

where κ is again an arbitrary integration constant and H is again given in the form of (413), with the following coefficients:

$$\begin{aligned} h_1 &= \frac{1}{\Delta}\left\{-\dot{\gamma}\,\delta + \delta^2 + \frac{\theta}{3}\gamma\,\delta + \gamma\,\dot{\delta} - \gamma^2 \right. \\ &\quad \left. \left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p) - \frac{(m - 3\epsilon)}{a^2}\right]\right\}, \\ h_2 &= \frac{1}{\Delta}\left\{-\dot{\alpha}\,\beta + \beta^2 + \frac{\theta}{3}\alpha\,\beta + \alpha\,\dot{\beta} - \alpha^2 \right. \\ &\quad \left. \left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p) - \frac{(m - 3\epsilon)}{a^2}\right]\right\}, \\ h_3 &= \frac{1}{2\Delta}\left\{-\dot{\alpha}\,\beta - \dot{\beta}\,\gamma - \beta\,\delta - \alpha\,\delta\left(\theta + \frac{\xi}{2}\right) + \beta\,\gamma\left(\frac{2}{3}\theta + \frac{\xi}{2}\right) \right. \\ &\quad \left. +\alpha\,\gamma\left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p) - \frac{(m - 3\epsilon)}{a^2}\right]\right\}. \end{aligned} \tag{453}$$

We are then able to perform the quantization by employing the standard method described by (423)–(428) and by making the same choice $\omega(t) = 0$ in order to decouple the first-order differential system, (428). We then find that

$$\begin{aligned} \alpha^4(t) &= \kappa^2 \, a^{10} \, e^{2\xi t}\left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) \right. \\ &\quad \left. +\frac{1}{2}(\rho + p) - \frac{(m - 3\epsilon)}{a^2}\right]^{-1}, \\ \beta(t) &\equiv \gamma(t) = 0, \\ \delta(t) &= \frac{\kappa}{\alpha(t)} \, a(t)^5 \, e^{\xi t}. \end{aligned} \tag{454}$$

The coefficients of the Hamiltonian for the choice given by (454) are

$$\begin{aligned} h_1(t) &= \frac{\kappa}{\alpha^2(t)} \, a^5 \, e^{\xi t} = -h_2(t), \\ h_3(t) &= \frac{1}{2}\left(\frac{\dot{\alpha}}{\alpha} - \theta - \frac{\xi}{2}\right). \end{aligned} \tag{455}$$

The decoupled first-order differential system then yields the following results:

$$\begin{aligned} r\,\cos(2\varphi) &= \ln\left(b \, \frac{a^3}{\alpha} \, e^{\frac{\xi}{2}t}\right), \\ r\,\sin(2\varphi) &= -\int_t\left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p) \right. \\ &\quad \left. -\frac{(m - 3\epsilon)}{a^2}\right]^{1/2} dt', \end{aligned} \tag{456}$$

where b is an integration constant. We then have that

$$\begin{aligned} E &= \frac{1}{\alpha} \, Q = \frac{1}{\sqrt{2}\alpha}(\hat{a} + \hat{a}^\dagger), \\ H &= \frac{\alpha}{\kappa} \, a^{-5} \, e^{-\xi t} \, P = -i \, \frac{\alpha}{\sqrt{2}} \, a^{-5} \, e^{-\xi t}(\hat{a} - \hat{a}^\dagger). \end{aligned} \tag{457}$$

and therefore the commutator between the above observables is given by the expression

$$[E, H] = i \, a(t)^{-5} \, \frac{e^{-\xi \, t}}{\kappa}. \tag{458}$$

Finally, the total noises for E and H are related to the total noises for the vacuum state in the same way as before:

$$\begin{aligned} \left\langle\Psi||\Delta E|^2|\Psi\right\rangle &= \cosh(2r) \left\langle 0||\Delta E|^2|0\right\rangle, \\ \left\langle\Psi||\Delta H|^2|\Psi\right\rangle &= \cosh(2r) \left\langle 0||\Delta H|^2|0\right\rangle, \end{aligned} \tag{459}$$

where the function r is given by (456).

From (459), it follows that the total noise at a time t is always greater than its vacuum value and that it increases with r.

3.8 Milne Background

A particular class of the FLRW geometries dealt with in this section merits explicit attention. This is the case analyzed by Milne, which contains a portion of Minkowski geometry. The metric is then FLRW-type, where the radius of the universe, the three-curvature and the expansion are given by the equalities

$$\begin{aligned} a(t) &= t, \\ \epsilon &= +1, \\ \theta &= \frac{3}{t}. \end{aligned} \tag{460}$$

respectively.

We will present only the following results:

3.8.1 Scalar Perturbations

In the case of scalar perturbations, the vorticity should vanish, which implies that the magnetic part of Weyl conformal tensor will also be zero; thus, we have that

$$\begin{aligned} \delta\omega_{ij} &= 0, \\ \delta H_{ij} &= 0. \end{aligned} \tag{461}$$

With the notations used before, the other perturbed quantities are listed below:

Geometric Quantity:

$$\delta E_{ij} = E(t)\hat{Q}_{ij}(\vec{x}).$$

Kinematic Quantities:

$$\begin{aligned} \delta V_0 &= -\delta V^0 = \frac{1}{2}\delta g_{00} = \frac{1}{2}\beta(t)Q(\vec{x}) + \frac{1}{2}Y(t), \\ \delta V_k &= V(t)Q_k(\vec{x}), \\ \delta a_k &= \Psi(t)Q_k(\vec{x}), \\ \delta\sigma_{ij} &= \Sigma(t)\hat{Q}_{ij}(\vec{x}), \\ \delta\theta &= B(t)Q(\vec{x}) + Z(t). \end{aligned}$$

Matter Quantities:

$$\begin{aligned} \delta\rho &= N(t)Q(\vec{x}) + L(t), \\ \delta\pi_{ij} &= \xi\delta\sigma_{ij} = \xi\Sigma(t)\hat{Q}_{ij}(\vec{x}), \\ \delta p &= \lambda\delta\rho, \\ \delta q_k &= q(t)Q_k(\vec{x}), \end{aligned}$$

where we have used again the proportionality relation between the perturbed anisotropic pressure and the shear; we have also considered the standard formulation, in which the perturbed pressure is proportional to the density. The quantity $\beta(t)$ is gauge-dependent and $Y(t)$, $Z(t)$ and $L(t)$ are homogeneous terms.

With help of the quasi-Maxwellian equations, we obtain the following system for the above quantities:

$$\dot{E} = -\frac{\xi}{2} + \frac{\theta}{3}E + \frac{\xi\theta}{6}\Sigma + \frac{m}{2}q = 0, \tag{462}$$

$$\frac{2\theta^2}{3}\left(\frac{1}{3} + \frac{\epsilon}{m}\right)\left[E - \frac{\xi}{2}\Sigma\right] + N + \theta q = 0, \tag{463}$$

$$\dot{B} + \frac{2\theta}{3}B + \frac{\theta^2}{6}\beta(t) + \frac{\theta^2}{9}m\Psi + \frac{(1+3\lambda)}{2}N = 0, \tag{464}$$

$$\dot{\Sigma} + E + \frac{\xi}{2}\Sigma - m\Psi = 0, \tag{465}$$

$$V = \left(\frac{1}{3} + \frac{\epsilon}{m}\right)\Sigma - \frac{3}{\theta^2}B - \frac{9}{2\theta^2}q, \tag{466}$$

$$\dot{N} + (1+\lambda)\theta N - \frac{\theta^2}{9}q = 0, \tag{467}$$

and

$$\dot{q} + \theta q - \lambda N - \frac{2\xi\theta^2}{9}\left(\frac{1}{3} + \frac{\epsilon}{m}\right)\Sigma = 0. \tag{468}$$

The dynamical equations for the homogeneous terms $Z(t)$ and $L(t)$ are written as follows:

$$\dot{Z} + \frac{2\theta}{3}Z + \frac{(1+3\lambda)}{2}L + \frac{\theta^2}{6}Y = 0, \tag{469}$$

and

$$\dot{L} + (1+\lambda)\theta L = 0. \tag{470}$$

Let us solve this system for the special simple case where $q = 0$. From (468), we then have the dynamical equation for q:

$$-\lambda N - \frac{2\xi\theta^2}{9}\left(\frac{1}{3} + \frac{\epsilon}{m}\right)\Sigma = 0. \tag{471}$$

Equations (460) and (467) give

$$N(t) = N_0 t^{-3(1+\lambda)}, \tag{472}$$

where N_0 is a constant.

From (471) and (472), we obtain the result

$$\Sigma(t) = -\frac{\lambda N_0}{2\xi}\left(\frac{1}{3} + \frac{\epsilon}{m}\right)^{-1} t^{-(1+3\lambda)}. \tag{473}$$

These results substituted in (463), we find that

$$E(t) = -\frac{N_0}{6}\left(1 + \frac{3\lambda}{2}\right)\left(\frac{1}{3} + \frac{\epsilon}{m}\right)^{-1} t^{-(1+3\lambda)}. \tag{474}$$

Equation (462) becomes automatically valid if we use the above results for $N(t)$, $\Sigma(t)$ and $E(t)$. Equation (465) then gives $\Psi(t)$ as

$$\Psi(t) = \frac{N_0}{2m}\left(\frac{1}{3} + \frac{\epsilon}{m}\right)^{-1}\left[\frac{\lambda(1+3\lambda)}{\xi}t^{-1} - \frac{(2+9\lambda)}{6}\right]t^{-(1+3\lambda)}. \tag{475}$$

The constant N_0 cannot be zero, since the result would then be trivial. Equations (464) and (466) determine the quantities $B(t)$ and $V(t)$ in terms of $N(t)$, $\Psi(t)$ and $\Sigma(t)$, $B(t)$, respectively. Both quantities may be obtained if the gauge-dependent function $\beta(t)$ is chosen. They are therefore "bad" quantities. The minimal closed set of quantities for perturbations in Milne universe is

$$\mathcal{M}^{scalar}_{[A]} = \{E, \Sigma, N, \Psi\}.$$

The homogeneous part of $(\delta\rho)$, $L(t)$, is directly determined by (469):

$$L(t) = L_0 t^{-3(1+\lambda)}, \tag{476}$$

where again L_0 denotes a constant.

The function $Z(t)$, whose dynamics is given by (469), can only be integrated by choosing another homogeneous term $(Y(t))$. That completes the solution for $q = 0$.

We can analyze the behavior of the above solution for different values of λ. The results are as follows:

1. $\lambda > -\frac{1}{3}$:

   E, Σ, N, and Ψ go to zero when $t \longrightarrow \infty$;

2. $\lambda = -\frac{1}{3}$:

   E, Σ, andΨ are constant; Ngoes to zero when $t \longrightarrow \infty$;

3. $-1 < \lambda < -\frac{1}{3}$:

   E, Σ, and Ψ diverge when $t \longrightarrow \infty$and Ngoes to zero;

4. $\lambda = -1$ (vacuum λ):

   E, Σ, and Ψ diverge when $t \longrightarrow \infty$ and N is constant;

5. $\lambda < -1$ (unphysical situation):

   E, Σ, N, and Ψ diverge when$t \longrightarrow \infty$

3.8.2 Vector Perturbations

In this case, the original dynamical system, (354a–354e)–(355a–355c), yields

$$\begin{aligned}
&\dot{E} - \frac{\xi}{2}\dot{\Sigma} + \frac{2}{3}\theta E + \frac{1}{2a^2}(m - 2\epsilon) H + \frac{1}{4} q = 0,\\
&\dot{\Sigma} + \left(\frac{\theta}{3} + \frac{\xi}{2}\right)\Sigma + E - \frac{1}{2}\Psi = 0,\\
&\dot{\Omega} + \frac{\theta}{3}\Omega + \frac{1}{2}\Psi = 0,\\
&\dot{H} + \frac{\theta}{3} H - \frac{1}{2} E - \frac{\xi}{4}\Sigma = 0,\\
&\dot{q} + \frac{4}{3}\theta q + \frac{1}{a^2}(m + 2\epsilon)\xi\Sigma = 0,
\end{aligned} \tag{477}$$

and

$$\begin{aligned}
&\Sigma + \Omega + 2H = 0,\\
&E - \frac{\xi}{2}\Sigma + \frac{2}{3}\theta H = 0,\\
&\frac{1}{a^2}(m + 2\epsilon) H + \frac{1}{2} q = 0.
\end{aligned} \tag{478}$$

We will present here only the three cases dealt with in Section 3.5.4: the isotropic, irrotational and Stokesian fluids. The results are as follows:

Isotropic Model: For $\Sigma = 0$, we obtain

$$\begin{aligned}
E(t) &= \mu t^{-2}\\
H(t) &= \frac{\mu}{2} t^{-1}\\
\Psi(t) &= 2\mu t^{-2}\\
q(t) &= -(m + 2)\mu t^{-3},
\end{aligned} \tag{479}$$

where μ is an integration constant and we have used $\epsilon = +1$. These functions of t diverge when $t \to 0$ and become null for $t \to \infty$.

Irrotational Model: For $\Omega = 0$, the acceleration Ψ is also zero and

$$\begin{aligned}
\Sigma(t) &= \nu t^{-2} e^{-\xi t},\\
E(t) &= \nu e^{-\xi t} t^{-2}\left(\frac{1}{t} + \frac{\xi}{2}\right),\\
H(t) &= -\frac{\nu}{2} t^{-2} e^{-\xi t},\\
q(t) &= \nu (m + 2) t^{-4} e^{-\xi t}.
\end{aligned} \tag{480}$$

These functions also diverge when $t \to 0$ and become zero when $t \to \infty$.

Stokesian Fluid: If we consider $q = 0$, the only possible solution is trivially zero. We conclude therefore that vector perturbations in Milne universes must have a nonzero heat flux.

3.8.3 Tensor Perturbations

The original (393a–393c) yield a closed dynamical system in the variable (E, Σ):

$$\begin{aligned}
&\dot{E} + \left(\theta + \frac{\xi}{2}\right) E + \left[\frac{\xi}{2}\left(\frac{\theta}{3} + \frac{\xi}{2}\right) - \frac{1}{a^2}(m - 3\epsilon)\right]\Sigma = 0,\\
&\dot{\Sigma} + \left(\frac{2}{3}\theta + \frac{\xi}{2}\right)\Sigma + E = 0,
\end{aligned} \tag{481}$$

where H is given by the constraint

$$\Sigma = H.$$

We then have the following set of *good* quantities for tensorial perturbations in the Milne background:

$$\mathcal{M}_A = \{E, H\}.$$

3.9 WIST Model: Scalar Perturbations

This section deals with the dynamical system of the perturbed quantities that are relevant for complete knowledge of the system. The resulting equations fully describe the perturbation evolution, according to the quasi-Maxwellian equations of gravitation.

In this case, we will assume that the background of the model can be characterized by a source consisting of a scalar field minimally coupled to the gravitational field. The energy-momentum tensor for a minimal-coupling scalar field is represented by a perfect fluid and it can be demonstrated that the general linear perturbations of this fluid also behave as a perfect fluid. This property of the source simplifies the equations and the minimal set of observables determining the scalar linear perturbations of the model can be obtained from the following equations:

$$\left(E^{\alpha\beta}\right)^{\cdot}+E^{\alpha\beta}-\frac{3}{2}E^{\mu(\alpha}\sigma^{\beta)}{}_{\mu}+h^{\alpha\beta}=-\frac{1}{2}(p+\rho)\sigma^{\alpha\beta}, \quad (482)$$

$$(\delta\sigma_{\mu\nu})^{\cdot}+\frac{1}{3}h_{\mu\nu}(\delta a^{\alpha})_{;\alpha}-\frac{1}{2}\delta a_{(\mu;\nu)}+\frac{2}{3}\theta\delta\sigma_{\mu\nu}=-\delta E_{\mu\nu}, \quad (483)$$

and

$$(p+\rho)\delta a_{\mu}=(\delta p)_{,\beta}h_{\mu}^{\beta}-p_{,\beta}\delta(h_{\mu}^{\beta}). \quad (484)$$

The equations are considered to be linear in the perturbations; so, to solve them, we will split the perturbations on the scalar spherical harmonics basis defined by (273) in terms of the conformal scale factor $a(\eta)$. Since the model we are investigating has an open three-section, the eigenvalue m can assume the following values:

$$m=q^2+1, \qquad 0\leq q\leq\infty. \quad (485)$$

With the scalar function Q, we can construct the vector and tensor quantities $\hat{Q}_{\alpha}$ and $\hat{Q}_{\mu\nu}$, respectively. Using this base, we can expand the perturbations as follows:

$$\delta E_{\mu\nu}=\sum_q E_{(q)}\,{}^{(q)}\hat{Q}_{\mu\nu}, \quad (486)$$

$$\delta\sigma_{\mu\nu}=\sum_q \sigma_{(q)}\,{}^{(q)}\hat{Q}_{\mu\nu}, \quad (487)$$

$$\delta a_{\mu}=\sum_q \psi_{(q)}\,{}^{(q)}\hat{Q}_{\mu}, \quad (488)$$

and

$$\delta V_{\mu}=\sum_q V_{(q)}\,{}^{(q)}\hat{Q}_{\mu}. \quad (489)$$

3.9.1 Dynamics

From now on, we will suppress the indices q to simplify the notation. Substituting this decomposition in (482) and (483), after simple algebraic calculations (see Novello [122]), we obtain the following dynamical system for each mode of the variables E and Σ:

$$\Sigma'=\left[\frac{1}{\rho a}(3+m)-a\right]E, \quad (490)$$

and

$$E'=-\frac{a'}{a}E-\rho a\Sigma. \quad (491)$$

The prime denotes covariant derivative projected on V^{α}.

This dynamical system can be compactly written in matrix form as

$$\begin{pmatrix}\Sigma\\E\end{pmatrix}'=\mathcal{M}\begin{pmatrix}\Sigma\\E\end{pmatrix}, \quad (492)$$

The components of the matrix $\mathcal{M}$ are: $\mathcal{M}_{11}=0$; $\mathcal{M}_{12}=-a+(3+m)/(\rho a)$; $\mathcal{M}_{21}=-\rho a$; $\mathcal{M}_{22}=-a'/a$.

3.9.2 Hamiltonian Treatment

The examination of the perturbations of Robertson-Walker geometries, using the variables associated to the perturbed metric tensor $\delta g_{\mu\nu}$, admits a Hamiltonian formulation. In this vein, it was shown in detail by Novello [123] that the present formulation using variables E and Σ also admits a Hamiltonian formulation. The usual way to do this is to introduce auxiliary field variables as

$$\begin{pmatrix}P\\Q\end{pmatrix}=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}\Sigma\\E\end{pmatrix}. \quad (493)$$

The matrix S with components α,β,γ and δ is univocally defined up to canonical transformations.

In matrix notation, in terms of the auxiliary variables, the dynamical system becomes

$$\begin{pmatrix}P\\Q\end{pmatrix}'=\Xi\begin{pmatrix}P\\Q\end{pmatrix}, \quad (494)$$

where

$$\Xi=S\mathcal{M}S^{-1}-S'S^{-1}. \quad (495)$$

The requirement that (Q,P) be canonical variables implies as necessary and sufficient condition that

$$tr\,\Xi=tr\mathcal{M}+\frac{\Delta'}{\Delta}=0, \quad (496)$$

where Δ is the determinant of S.

In our case, the Hamiltonian is given by the expression

$$H=\frac{\Xi_{21}}{2}P^2-\frac{\Xi_{12}}{2}Q^2-\frac{\Xi_{11}}{2}(PQ+QP). \quad (497)$$

The matrices $\mathcal{M}$ and S univocally determine H up to canonical transformations. This freedom can be used to simplify our analysis in each particular case.

The background model we will investigate is asymptotic flat in the limits of the conformal time $\eta \rightarrow \pm\infty$. Convenient canonical transformations can fix the functions of the matrix S so that in the limit $\eta \rightarrow -\infty$ we obtain, for each mode m, the Hamiltonian of an harmonic oscillator in a Minkowski space-time. With this choice, the Hamiltonian that describes the system reads

$$H = \frac{1}{2}P^2 + \frac{1}{2}\left[8\tanh(2\eta) + (m-7) + \frac{\Sigma_0}{a^4}\right]Q^2 + [\tanh(2\eta)+1](PQ+QP). \quad (498)$$

A simple calculation using Hamilton's equations then shows that

$$Q_m''(\eta) + [q^2 - 3(\tanh(2\eta)2 - 1)]Q_m(\eta) = 0, \quad (499)$$

and

$$w_m^2(\eta) = q^2 - V_{eff}. \quad (500)$$

The exact solution to (499) is given by the expression

$$Q_m(\eta) = AF(a_1, b_1, c_1, z) + B\sinh(2\eta)F(a_2, b_2, c_2, z)], \quad (501)$$

where $F(a_1, b_1, c_1, z)$ and $F(a_2, b_2, c_2, z)$ are hypergeometric functions with the parameters

$$a_1 = \frac{3}{4} + \frac{I}{4}\sqrt{m-1}, a_2 = \frac{5}{4} + \frac{I}{4}\sqrt{m-1},$$
$$b_1 = \frac{3}{4} - \frac{I}{4}\sqrt{m-1}, b_2 = \frac{5}{4} - \frac{I}{4}\sqrt{m-1},$$
$$c_1 = \frac{1}{2}, \qquad c_2 = \frac{3}{2}. \quad (502)$$

The variable z is given by the expression

$$z = -\sinh^2(2\eta).$$

In the asymptotic limits $\eta \rightarrow \pm\infty$, we have that

$$Q_m^{in}(\eta) = \lim_{\eta\rightarrow-\infty} Q_m(\eta) = \frac{1}{\sqrt{w_m}}e^{-iq\eta}, \quad (503)$$

and

$$Q_m^{out}(\eta) = \lim_{\eta\rightarrow\infty} Q_m(\eta) = d_1(m)e^{-iq\eta} + d_2(m)e^{iq\eta}. \quad (504)$$

The expression for the amplitudes $d_1(m)$ and $d_2(m)$ in terms of trigonometric and gamma functions is

$$d_1 = \frac{1}{8}\frac{\sqrt{q}(q^2+1)\Gamma\left(\frac{iq}{2}\right)^2 \sinh(\pi q/2)}{\Gamma\left(\frac{3}{2}+\frac{iq}{2}\right)^2\left[\sin(\pi(1+iq)/4)^2 - \sin(\pi(1-iq)/4)^2\right]},$$
$$d_2 = \frac{-2+\cos(\pi(5+iq)/4)^2 + \cos(\pi(3+iq)/4)^2}{\sqrt{q}\left[\cos(\pi(5+iq)/4)^2 - \cos(\pi(3+iq)/4)^2\right]} \quad (505)$$

Using the scalar basis $l_m(x)$, the conjugate variables Q and P, which describe our dynamical system, can be expanded in terms of traveling waves according to the following equations:

$$Q(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}[Q_m^*(\eta)l_m^*(\mathbf{x}) + Q_m(\eta)l_m(\mathbf{x})], \quad (506)$$

and

$$P(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}[P_m^*(\eta)l_m^*(\mathbf{x}) + P_m(\eta)l_m(\mathbf{x})]. \quad (507)$$

3.9.3 Quantum Treatment of the Perturbations

The conjugate variables $Q(x)$ and $P(x)$ can be quantized, following standard procedures, by transforming the mode functions into operators (cf. Birrel [18]):

$$\hat{Q}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}\left[\hat{a}_m^- Q_m^* l_m^*(\mathbf{x}) + \hat{a}_m^+ Q_m l_m(\mathbf{x})\right]$$
$$\hat{P}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}\left[\hat{a}_m^- P_m^* l_m^*(\mathbf{x}) + \hat{a}_m^+ P_m l_m(\mathbf{x})\right]. \quad (508)$$

The classical canonical variables are replaced by operators $\hat{P}$ and $\hat{Q}$ satisfying the commutation relations $[\hat{Q}_\alpha, \hat{P}_\beta] = i\hbar\delta_{\alpha\beta}$ and $[\hat{Q}_\alpha, \hat{Q}_\beta] = [\hat{P}_\alpha, \hat{P}_\beta] = 0$, after which we can define the creation and annihilation operators:

$$\hat{a}_\alpha^\pm = \sqrt{\frac{\omega_\alpha}{2}}(\hat{Q}_\alpha(t) \mp \hat{P}_\alpha), \quad (509)$$

with the new commutation relations $[\hat{a}_\alpha^-, \hat{a}_\beta^+] = \delta_{\alpha\beta}$ and $[\hat{a}_\alpha^-, \hat{a}_\beta^-] = [\hat{a}_\alpha^+, \hat{a}_\beta^+] = 0$.

Given the expansion (506), we can write two different sets of functions corresponding to ingoing and outgoing waves, given by

$$\hat{Q}^{in}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}\left[\hat{a}_m^- v_m^* l_m^*(\mathbf{x}) + \hat{a}_m^+ v_m l_m(\mathbf{x})\right] \quad (510)$$

and

$$\hat{Q}^{out}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\,\mathbf{m}}{(2\pi)^{3/2}}[\hat{b}_m^- u_m^* l_m^*(\mathbf{x}) + \hat{b}_m^+ u_m l_m(\mathbf{x})], \quad (511)$$

respectively.

The operators $\hat{a}_m$ and $\hat{b}_m$ are related by Bogolyubov coefficients α_m and β_m,

$$\hat{b}_m = \alpha_m\hat{a}_m + \beta *_m \hat{a}_{-m}, \quad (512)$$

which coefficients satisfy the normalization condition $|\alpha_m|^2 - |\beta_m|^2 = 1$.

The two sets of isotropic mode functions u_m and v_m form a base of *in* and *out* functions. We can write this base, according to (503) and (504), as follows:

$$v_q(\eta) = \frac{1}{\sqrt{w_m}} e^{-I w_m \eta}, \tag{513}$$

and

$$u_q(\eta) = d_1(q)e^{-iq\eta} + d_2(q)e^{iq\eta}. \tag{514}$$

The "in" vacuum is described by the standard Minkowski mode function and the mode corresponding to "out" vacuum can be written as a linear combination of the basis v_q:

$$u_q = \alpha_q v_q - \beta^*{}_q v^*{}_q. \tag{515}$$

3.10 Nonlinear Electrodynamics: Scalar Perturbations

Singularities appear to be inherent to most physically relevant solutions of Einstein equations, in particular to all black hole and conventional cosmological solutions (see Hawking and Ellis [64]) known to this date. In the case of black holes, certain models have been proposed to avoid the singularity—cf. Bardeen [8], Barrabes and Frolov [10], Cabo and Ayon-Beato [27] and, Mars et al. [91]. Nonetheless, these models are not exact solutions of Einstein equations since there are no physical sources associated with them. Many attempts have tried to solve this problem by modifying general relativity, for instance, Cvetic [37], Tseytlin [156] and Horne and Horowitz [72]. More recently, it has been shown that in the framework of standard general relativity, it is possible to find spherically symmetric singularity-free solutions of the Einstein field equations that describe a regular black hole. The source of these solutions are generated by suitable nonlinear vector field Lagrangians, which in the weak-field approximation become the usual linear Maxwell theory demonstrated by Ayon-Beato and Garcia ([5–7]). Similarly, in Cosmology, many nonsingular cosmological models with bounce were constructed that violated the energy conditions or validity of Einstein gravity (see Section 2.4.2 for details).

In 2002, de Lorenci et al. investigated a cosmological model with a source produced by a nonlinear generalization of electrodynamics and succeeded to obtain a regular cosmological model. The Lagrangian of such model is a function of the field invariants up to second order—(85). This modification is expected to be relevant when the fields reach large values, as in the primeval era of our universe. The model lies in the framework of the Einstein field equations and the bounce is possible because the singularity theorems are circumvented by the appearance of a negative pressure, although the energy density is positive definite. Recently, a few papers started a detailed investigation of the transition from contraction to expansion in the bounces of several models (see Cartier et al. [32]).

In particular, in Einstein general relativity, models with stress-energy sources constituted by a collection of perfect fluids and FLRW-like geometry were examined by Peter and Pinto-Neto [131]. That paper claims that a generic result concerning the behavior of scalar adiabatic perturbations was obtained. The result is the following: scalar adiabatic perturbations can grow without limit in two situations represented by the points where the scale factor attains its minimum value and where $\rho + p = 0$. The first point corresponds to the moment at which the Universe passes through the bounce; the second corresponds to the transition from the region where the null energy condition (NEC) is violated to the region where it is not. However, these instabilities are not an intrinsic property of generic models with bounce, but a consequence of the existence of a divergence already in the background solution, when the source is described as a perfect fluid. We next show a specific example of a model with bounce, generated by a source representing two noninteracting perfect fluids, that has regular perturbations.

3.10.1 The Model

We set the fundamental line element as an FLRW metric. According to the definitions in Section 2.4.2, we rewrite the Einstein equations and the equation of energy conservation written for this metric as follows:

$$\left(\frac{\dot a}{a}\right)^2 + \frac{\epsilon}{a^2} - \frac{1}{3}\rho_\gamma = 0, \tag{516}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{\epsilon}{a^2} + p_\gamma = 0, \tag{517}$$

and

$$\dot\rho_\gamma + 3(\rho_\gamma + p_\gamma)\frac{\dot a}{a} = 0. \tag{518}$$

Insertion of (87) and (88) in (518) yields (89) for the magnetic field, where H_o is an arbitrary constant. With this result, (516) can be integrated. For $\epsilon = 0$, the solution is given by (94).

The interpretation of the source as a one-component perfect fluid in an adiabatic regime raises difficulties (see Peter and Pinto-Neto [131]). The sound velocity of the fluid in this case is given by the equality

$$\left(\frac{\partial p_\gamma}{\partial \rho_\gamma}\right) = \frac{\dot p_\gamma}{\dot\rho_\gamma} = -\frac{\dot p_\gamma}{\theta(\rho_\gamma + p_\gamma)}. \tag{519}$$

This expression, involving only the background, is not defined at those points where the energy density attains an extremum given by $\theta = 0$ or $\rho_\gamma + p_\gamma = 0$. In terms of the cosmological time, they are determined by $t = 0$ and $\pm tc = 12\alpha/kc^2$. These points are well-behaved regular points of the geometry, which indicates that the description of the source is not appropriate. This difficulty can be circumvented if one adopts another description for the source of the model. This can be achieved by separating the part of the source related to Maxwell dynamics from the additional nonlinear term dependent on a on the Lagrangian. The source then automatically splits in two noninteracting perfect fluids:

$$T_{\mu\nu} = T^1_{\mu\nu} + T^2_{\mu\nu}, \tag{520}$$

where

$$T^1_{\mu\nu} = (\rho_1 + p_1)V_\mu V_\nu - p_1 g_{\mu\nu}, \tag{521}$$

$$T^2_{\mu\nu} = (\rho_2 + p_2)V_\mu V_\nu - p_2 g_{\mu\nu}, \tag{522}$$

and

$$\rho_1 = \frac{1}{2}H^2, \tag{523}$$

$$p_1 = \frac{1}{6}H^2, \tag{524}$$

$$\rho_2 = -4\alpha H^4, \tag{525}$$

and

$$p_2 = -\frac{20}{3}\alpha H^4. \tag{526}$$

Using the above decomposition, it follows that each one of the two components of the fluid independently satisfies (518). This indicates that the source can be described by two noninteracting perfect fluids with equation of states $p_1 = \rho_1/3$ and $p_2 = 5\rho_2/3$. The equation of state for the second fluid should be understood only formally as a mathematical device to allow a fluid description.

3.10.2 Gauge Invariant Treatment of Perturbation

The source of the background geometry is represented by two fluids, each having an independent equation of state relating the pressure and the energy density. Following standard procedure, we consider arbitrary perturbations that preserve each equation of state. Thus, the general form of the perturbed energy-momentum tensor is written in the form

$$\delta T^i_{\mu\nu} = (1+\lambda_i)\delta(\rho_i V_\mu V_\nu) - \lambda_i \delta(\rho_i g_{\mu\nu}). \tag{527}$$

The background geometry is conformally flat. Therefore, any perturbation of the Weyl tensor is a true perturbation of the gravitational field. It is convenient to represent the Weyl tensor $W_{\alpha\beta\mu\nu}$ in terms of its corresponding electric $E_{\mu\nu}$ and magnetic $H_{\mu\nu}$ parts because these variables have the advantage that their perturbations are gauge invariant, since they are null in the background (Hawking [65]).

Since the equations of motion for the first-order perturbations are linear, it is useful to develop all perturbed quantities on the spherical-harmonics basis. Here, we will limit our analysis to perturbations represented on the scalar basis, which we also take from Section 3.4.

In the case of scalar perturbations, the fundamental set of equations determining the dynamics of the perturbations are

$$\left(\delta E_1^{\mu\nu}\right)^\bullet h_\mu{}^\alpha h_\nu{}^\beta + \left(\delta E_2^{\mu\nu}\right)^\bullet h_\mu{}^\alpha h_\nu{}^\beta + \theta\left(\delta E_1^{\alpha\beta} + \delta E_2^{\alpha\beta}\right) = -\frac{1}{2}(\rho_1 + p_1)\,\delta\sigma_1^{\alpha\beta} - \frac{1}{2}(\rho_2 + p_2)\,\delta\sigma_2^{\alpha\beta}, \tag{528}$$

$$(\delta E^1_{\alpha\mu} + \delta E^2_{\alpha\mu})_{;\nu} h^{\alpha\varepsilon} h^{\mu\nu} = \frac{1}{3}(\delta\rho_1 + \delta\rho_2)_{,\alpha} h^{\alpha\varepsilon} - \frac{1}{3}\dot{\rho}_1 \delta v_1^\varepsilon - \frac{1}{3}\dot{\rho}_2 \delta v_2^\varepsilon, \tag{529}$$

$$\left(\delta\sigma^1_{\mu\nu}\right)^\bullet + \left(\delta\sigma^2_{\mu\nu}\right)^\bullet + \frac{1}{3}h_{\mu\nu}(\delta a_1^\alpha + \delta a_2^\alpha)_{;\alpha} - \frac{1}{2}\left(\delta a^1_{\alpha;\beta} + \delta a^2_{\alpha;\beta}\right) h_{(\mu}{}^\alpha h_{\nu)}{}^\beta + \frac{2}{3}\theta\left(\delta\sigma^1_{\mu\nu} + \delta\sigma^2_{\mu\nu}\right) = -\delta E^1_{\mu\nu} - \delta E^2_{\mu\nu}, \tag{530}$$

$$-\lambda_1 \delta\left(\rho_{,\beta} h^\beta{}_\mu\right) + (1+\lambda_1)\rho\delta - a^1_\mu = 0, \tag{531}$$

$$-\lambda_2 \delta\left(\rho_{,\beta} h^\beta{}_\mu\right) + (1+\lambda_2)\rho\delta - a^2_\mu = 0. \tag{532}$$

The acceleration a^μ, the expansion θ and the shear $\sigma_{\mu\nu}$ in the above equations are parts of the irreducible components of the covariant derivative of the velocity field.

The expansion of the perturbations in terms of the spherical harmonic basis is given by the equations[30]

$$\delta\rho = N(t)Q, \tag{533}$$

$$\delta V^\mu = V(t) h^{\mu\alpha} Q_{,\alpha}, \tag{534}$$

$$\delta a^\mu = \dot{V}\, h^{\mu\alpha} Q_{,\alpha}, \tag{535}$$

$$\delta E^{\mu\nu} = E(t) P^{\mu\nu}, \tag{536}$$

$$\delta\sigma^{\mu\nu} = \Sigma(t) P^{\mu\nu}, \tag{537}$$

3.10.3 Dynamics

After presenting the necessary formalism, we shall start to analyze the perturbations of the previously described

[30] Since we are dealing with a linear process, each mode can be analyzed separately.

bouncing cosmological model. Using the above expansion in (528)–(532), we obtain the following results:

$$E_1 + E_2 = \frac{a^2}{6\epsilon + k^2}\left(N_1 + N_2 - \dot{\rho}_1 V_1 - \dot{\rho}_2 V_2\right), \quad (538)$$

$$\dot{E}_1 + \dot{E}_2 + \frac{1}{3}\theta\left(E_1 + E_2\right) = -\left(\frac{1+\lambda_1}{2}\right)\rho_1\Sigma_1 - \left(\frac{1+\lambda_1}{2}\right)\rho_2\Sigma_2, \quad (539)$$

$$\dot{\Sigma}_1 + \dot{\Sigma}_2 - \dot{V}_1 - \dot{V}_2 = -E_1 - E_2, \quad (540)$$

$$-\lambda_1\left(N_1 - \dot{\rho}_1 V_1\right) + \left(1+\lambda_1\right)\rho_1\dot{V}_1 = 0, \quad (541)$$

and

$$-\lambda_2\left(N_2 - \dot{\rho}_2 V_2\right) + \left(1+\lambda_2\right)\rho_2\dot{V}_2 = 0. \quad (542)$$

These equations can be rewritten in a more convenient way as follows:

$$\dot{\Sigma}_1 = -\left(\frac{2\lambda_1(3\epsilon + k^2)}{a^2(1+\lambda_1)\rho_1} + 1\right)E_1, \quad (543)$$

$$\dot{\Sigma}_2 = -\left(\frac{2\lambda_1(3\epsilon + k^2)}{a^2(1+\lambda_2)\rho_2} + 1\right)E_2, \quad (544)$$

$$\dot{E}_1 + \frac{1}{3}\theta E_1 = -\frac{1}{2}\left(1+\lambda_1\right)\rho_1\Sigma_1, \quad (545)$$

and

$$\dot{E}_2 + \frac{1}{3}\theta E_2 = -\frac{1}{2}\left(1+\lambda_2\right)\rho_2\Sigma_2, \quad (546)$$

The whole set of scalar perturbations can be expressed in terms of the two basic variables: E_i and Σ_i. The corresponding equations can be decoupled. The result in terms of the variables E_i is the following:

$$\ddot{E}_i + \frac{4+3\lambda_i}{3}\theta\dot{E}_i + \left[\frac{2+3\lambda_i}{9}\theta^2 - \left(\frac{2}{3}+\lambda_i\right)\rho_i - \frac{1}{6}(1+3\lambda_j)\rho_j - \frac{(3\epsilon+k^2)\lambda_i}{a^2}\right]E_i = 0. \quad (547)$$

There is no sum over indices and $j \neq i$ in this expression. In our case, λ_i can take the values $\lambda_i = \left(\frac{1}{3}, \frac{5}{3}\right)$. In the first alternative, the equation for E_1 becomes

$$\ddot{E}_1 + \frac{5}{3}\theta\dot{E}_1 + \left[\frac{1}{3}\theta^2 - \rho_1 - \rho_2 - \frac{5k^2}{3a^2}\right]E_{(1)} = 0. \quad (548)$$

We should analyze the behavior of these perturbations in the neighborhood of the points where the energy density attains an extremum. This means not only the bouncing point, but also the point at which $\rho + p$ vanishes. Let us start by examining the bouncing point $t = 0$.

If we consider up to second-order terms in perturbation theory, the expansion of the equation for E_1 in the neighborhood of the bouncing is given by the expression:

$$\ddot{E}_1 + a_c t\dot{E}_1 + (b + b_1 t^2)E_1 = 0, \quad (549)$$

where the constant a_c and the parameters b and b_1 are defined as follows:

$$a_c = \frac{5}{2t_c^2}, \quad (550)$$

$$b = -\frac{m^2}{\sqrt{6}H_0 t_c}, \quad (551)$$

and

$$b_1 = -\frac{b}{2t_c^2} - \frac{3}{4t_c^4}. \quad (552)$$

Defining a new variable f as

$$f(t) = E_1(t)\exp\left[\left(\frac{a_c}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a_c^2}{4}}\right)t^2\right], \quad (553)$$

and transforming the time coordinate as follows

$$\xi = -i\left(\sqrt{b_1 - \frac{a_c^2}{4}}\right)t^2, \quad (554)$$

we obtain the following confluent hypergeometric equation (cf. Abramowitz and Stegun [1])

$$\xi\ddot{f} + (1/2 - \xi)\dot{f} + ef = 0, \quad (555)$$

where

$$e = \frac{i(b - a_c/2)}{4(b_1 - a_c^2/4)^{1/2}} - \frac{1}{2}. \quad (556)$$

The solution to this equation is

$$f(t) = f_o\,\mathcal{M}\left[d, 1/2, -i\left(\sqrt{4b_1 - a_c^2}\right)\frac{t^2}{2}\right], \quad (557)$$

where f_o is an arbitrary constant and $\mathcal{M}[d, 1/2, \xi]$ is a confluent hypergeometric function.

The confluent hypergeometric function is well behaved in this neighborhood and so is the perturbation $E_1(t)$, given by the equation

$$E_1(t) = \Re\left\{f_o\,\boldsymbol{\mathcal{M}}\left[d, 1/2, -i\left(\sqrt{4b_1 - a_c^2}\right)\frac{t^2}{2}\right]\exp\left[\left(-\frac{a_c}{4} + \frac{i}{2}\sqrt{b_1 - \frac{a_c^2}{4}}\right)t^2\right]\right\}. \quad (558)$$

For the perturbation E_2 in the same neighborhood, the procedure we followed before results in the following equation:

$$\ddot{E}_2 + a_c t\dot{E}_2 + \left(b + b_1 t^2\right)E_2 = 0. \quad (559)$$

This is the same equation we obtained for E_1, except for the values of the parameters a_c, b and b_1, which in this case are

$$a_c = \frac{9}{2t_c^2}, \tag{560}$$

$$b = \frac{3}{2t_c^2} - 5\frac{m^2}{\sqrt{6}H_0 t_c}, \tag{561}$$

and

$$b_1 = -\frac{5m^2}{t_c^3 H_0\sqrt{6}} - \frac{5}{t_c^4}. \tag{562}$$

The solution in this case is, then,

$$E_2(t) = \Re\left\{ f_o\,\mathcal{M}\left[d, 1/2, -i\left(\sqrt{4b_1 - a_c^2}\right)\frac{t^2}{2}\right] \exp\left[-\left(\frac{a_c}{4} - \frac{i}{2}\sqrt{b_1 - \frac{a_c^2}{4}}\right)t^2\right]\right\}. \tag{563}$$

Again, the confluent hypergeometric function is well behaved in this neighborhood and so is the perturbation $E_2(t)$. At the neighborhood of the point $t = t_c$, the equation for the perturbation E_1 is

$$\ddot{E}_1 + a_c\dot{E}_1 + (b + b_1 t)\,E_1 = 0, \tag{564}$$

with parameters a_c, b and b_1 now given by the equalities

$$a_c = \frac{5}{4t_c}, \tag{565}$$

$$b_= -\frac{3}{4t_c^2} - \frac{\sqrt{3}m^2}{6H_0 t_c}, \tag{566}$$

and

$$b_1 = \frac{\sqrt{3}}{4t_c^2}\left(\frac{m^2}{3H_0} - \frac{3}{2t_c}\right). \tag{567}$$

This equation differs from (549) and (559), which were obtained in the neighborhood of $t = 0$. To proceed, we transform the variable as follows:

$$E_1(t) = w(t)\exp\left(-\frac{a_c t}{2}\right). \tag{568}$$

The differential equation for the new variable is

$$\ddot{w} + (b - (a_c/2)^2 + b_1 t)w = 0. \tag{569}$$

The solution for this equation is

$$w(t) = \left[w_0\,\text{AiryAi}\left(-\frac{b - (a_c/2)^2 + b_1 t}{b^{2/3}}\right)\right]. \tag{570}$$

The AiryAi are regular well-behaved functions in this neighborhood and also the perturbations E_1.

Finally, we look for the equation for E_2 in the neighborhood of $t = t_0$ and it becomes

$$\ddot{E}_2 + a_c\dot{E}_2 + (b + b_1 t)\,E_2 = 0, \tag{571}$$

where the parameters a_c, b and b_1 are now

$$a = \frac{9}{4t_c}, \tag{572}$$

$$b = \frac{5}{t_c}\left(\frac{5}{4t_c} - \frac{\sqrt{3}m^2}{6H_0}\right), \tag{573}$$

and

$$b_1 = \frac{5\sqrt{3}}{2t_c^2}\left(\frac{1}{t_c} - \frac{m^2}{6H_0}\right). \tag{574}$$

This equation differs from (564) only in the numerical values of the parameters a_c, b and b_1. We therefore obtain the same regular solution

$$E_2 = \Re\left[\exp\left(-\frac{a_c t}{2}\right) w_0\,\text{AiryAi}\left(-\frac{b - (a_c/2)^2 + b_1 t}{b_1^{2/3}}\right)\right]. \tag{575}$$

To summarize, there recently has been renewed interest in nonsingular cosmology. In response, a few authors have argued against these models based on instability reasons. Peter and Pinto-Neto [131] have argued that a rather general analysis shows that there are instabilities associated to some special points of the geometrical configuration. They correspond to the bouncing points of the model and maxima of the energy density, where the description of the matter content in terms of a single perfect fluid fails to apply. In the present paper, we have shown, by direct analysis of a specific nonsingular universe, that the result claimed in the aforementioned paper does not apply to our model. We took the example from De Lorenci et al. [39], who showed that a nonlinear electrodynamic theory avoids the singularity. We have used the quasi-Maxwellian equations of motion—cf. Novello et al. [122–124]—to analyze the perturbed set of Einstein equations of motion. We have shown that in the neighborhood of the special points at which a change of regime occurs, all independent perturbed quantities are well behaved. Consequently, the model presents no difficulty associated with instability. This paves the way to investigate models with bounce in more detail and to consider them as good candidates to describe the evolution of the Universe.

4 On the Role of Initial Conditions to the Equivalence Theorem

In this section, we briefly discuss the explicit question that arises in perturbation theory: how do the Einstein equations determine the Weyl tensor, if they contain information only on the traces of the curvature tensor?

The Einstein equations relate the energy-momentum tensor with the traces of the curvature tensor (local quantities), leaving the remaining components of curvature tensor, which correspond to the Weyl tensor (a nonlocal quantity), undetermined. However, this indetermination is only apparent, because Bianchi's identities relate the traces of the curvature tensor with the Weyl tensor via (9). Notice, however, that this relation involves partial derivatives. Thus, substituting the Einstein equations into (9), we obtain the set of equations that involves the energy-momentum and Weyl tensors, which leaves the traces of the curvature out—see (10). These equations are direct consequences of the Einstein equations, but not equivalent to them, in principle. In order to establish the equivalence, it is necessary to impose an appropriate initial conditions—see details concerning Lichnerowicz's theorems in [88].

From the mathematical point of view, the problem is automatically solved. It is not solved, however, from the physical viewpoint because the mathematical solution calls for initial conditions that have to be determined from empirical data on a Cauchy surface. More specifically, the empirical data determine the initial conditions of the physical system that we want to describe taking into account the curvature and energy-momentum tensors. Actually, these empirical data are precisely the curvature tensor of space-time and the energy-momentum tensor. Since the curvature tensor has 20 independent components, we are faced with the following question: how can we determine the Cauchy data specifying the metric components g_{ij} and their first derivatives with respect to time $g_{ij,0}$ $(i, j = 1, 2, 3)$? The Einstein equations are differential equations for the metric potential $g_{\mu\nu}$; they give no information about the curvature tensor as an initial condition and then, this situation imposes serious ambiguity on the determination of boundary conditions. This ambiguity clearly appears in analyses of gravitational waves and can be consistently revealed even in simple instances of perturbation theory, in which the gravitational theory is linearized. Indeed, if we consider an exact solution with a perfect fluid as source of curvature, the perturbed equations describing gravitational waves reduce to

$$\delta R_{\mu\nu} = 0. \tag{576}$$

In the QM formalism, (576) yields

$$\delta W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2}(\rho + p)\delta V_{\mu;[\alpha} V_{\beta]}. \tag{577}$$

Equations (577) are not precisely equivalent to (576) because they consider distinct initial conditions. Besides, (577) has a kind of "source" for the gravitational field (in this case, due to coupling terms with shear tensor) that is absent in Lifshitz perturbation theory. To eliminate this ambiguity by means of clear examples, we have to resort to two recent papers on tensorial perturbations of isotropic metrics using the QM formalism [70, 133, 134]. The authors used null coordinates to analyze tensorial perturbations in the Robertson-Walker metrics and encountered a gravitational-wave type of solution different from the solutions obtained via the Bardeen method of gauge-invariant perturbations [9, 50]. The definition of gravitational waves in [70, 133, 134] leads to no restriction on the equation of state of the fluid on the isotropic background, the opposite of what was obtained by [17, 159], in which the sound speed is equal to the velocity of light. As we have explained, this difference can be associated with the different boundary conditions in each approach, which are fundamental to consistently define gravitational waves in each case. We will not discuss the two papers in detail, but we would like to emphasize the question: in a nonlinear regime of gravitational theory, how can we translate the information contained in the non-null Weyl tensor in terms of the perturbed potential δg_{ij}?

In other words, GR does not contemplate in its boundary conditions any kind of information about intrinsic accelerations of a given configuration in the manifold, which would lead to the development of a nonlocal description of the space-time, as suggested by [20, 33, 93, 94]. This type of information must explicitly be given to the QM formalism because these equations have partial derivatives of the energy-momentum tensor and contain higher-order derivatives of the metric tensor in the dynamics.

5 Concluding Remarks

During the entire history of the quasi-Maxwellian equations of general relativity, they have only been considered as an alternative approach, equivalent to the Einstein equations, although perhaps more complicated. However, even Jordan and his collaborators knew the importance of this issue, as evidenced by the following quote: *With this paper, we start an enterprise which, considering the present state of the scientific development of this field, is of some urgency. We want to collect and describe all those exact solutions which are dispersed in the literature and thus probably, in their completeness, known to very few authors only* [79, 80]. After all these years of their seminal work, we summarize here all the work that has been done in this area showing that this formalism is actually as powerful as the standard one, since the well-known solutions of GR are easily regained from QM-formalism.

In the case of conformally flat universes, whether singular [122] or not [120], this approach is more convenient to treat all types of perturbations (scalar, vectorial and tensorial). It reduces the dynamics to a pair of equations in Σ and E, which are physically meaningful variables providing a dynamical planar system and a reparametrization of these variables allows to establish a gauge-invariant Hamiltonian treatment for the perturbation. In particular, a few years ago, the interest on nonsingular cosmology was renewed. We showed that the corresponding models are completely stable, on the basis of results obtained from the QM-approach. In the case of the Schwarzschild metric [83] as well as in the Kasner case [100], it is also possible to use the Stewart Lemma [148] to apply this method and obtain significant results.

There are many areas of research on this topic. Some of them concern the development of deep analogies between QM-equations and the Maxwell theory [60, 61]. Another topic concerns modifications on the dynamics of general relativity suggested by Bianchi's identities for the Weyl tensor [108]. Most of these issues are still under development.

Acknowledgments The authors would like to thank all the participants of the Pequeno Seminário at CBPF for valuable contribution to certain parts of this review. We also thank CNPq, FAPERJ, CAPES (BEX 13956/13-2) FINEP for their financial support. MN acknowledge the ICRANet-Pescara staff for the hospitality during some stages of this work.

Appendix A: An Example of Further Developments in the QM-Formalism

This appendix presents an example in which the QM-formalism is modified and the analogy with the electromagnetism is strengthened. Modifications of this kind indicate alternative, more intuitive ways to solve problems appearing in general-relativity theory. We present an enlightening example next.

Electrodynamics is the paradigm of field theory. Its theoretical and experimental properties have been simulated and sought for in many other theories and in particular in the analysis of gravitational phenomena. Much work has been done along this line, which discusses the resemblance between electrodynamics and gravidynamics.[31] However, it seems possible to further improve this similarity, as we shall show.

In this vein, we review here a modification of Einstein's theory of general relativity under certain special states of the geometry of the space-time. Since the original proposal of Einstein's geometrization of gravitational processes, many physicists have discussed alternative models of gravitation. The kind of theory we shall analyze here is given by means of the metric properties—represented by a symmetrical metric tensor $g_{\mu\nu}(x)$—and by two other functions, $\varepsilon(x)$ and $\mu(x)$, which are independent of the metric,[32] but have intimate connection to the space-time.

For pedagogical reasons, we find it convenient to limit our considerations to the case in which both ε and μ are constants. The meaning one should attribute to these two constants comes from direct analogy with the dielectric and permeability constants of a given medium in electrodynamics.

We shall simplify the model by merely stating that ε and μ can be provisionally identified with the characteristics of certain states of tensions, in free space-time, due to an average procedure on (quantum) properties of gravitation.

In other words, ε and μ are interpreted as the result—in a macroscopic level—of some sort of averaging microscopic field fluctuation.[33] This is perhaps not difficult to assume if we can say exactly how the equations of motion of gravity phenomena must be modified by them, as we shall do later. We remark that we are not supporting this interpretation but merely suggesting it as a provisional *sursis* of the model.

We shall describe gravitational interaction[34] by means of a fourth-rank tensor $Q_{\alpha\beta\mu\nu}$. We shall set up its algebraic properties and give its dynamics. It is possible to separate this tensor, for an observer moving with four-velocity V^{μ}, into four second-order symmetric trace-free tensors $E_{\alpha\beta}$, $B_{\alpha\beta}$, $D_{\alpha\beta}$ and $H_{\alpha\beta}$. The principal result is then obtained by showing that it is possible to select a class of observers with velocity ℓ^{μ} in such a way as to have equations of motion for $Q_{\alpha\beta\mu\nu}$ similar to the Maxwell equations for the electrodynamics. That is, for $E_{\alpha\beta}$, $D_{\alpha\beta}$, $B_{\alpha\beta}$ and $H_{\alpha\beta}$ separated into two groups: one containing only $E_{\alpha\beta}$ and $B_{\alpha\beta}$ (and their derivatives) and the other containing only $H_{\alpha\beta}$ and $D_{\alpha\beta}$ (and their derivatives). These equations have the same formal structure of Maxwell's equations in a given general medium. We therefore come to the conclusion that the present theory has a class of privileged observers in which gravitational field equations admit this simple separated form. Any other observer, which is in motion with

[31] This resemblance is far from being accepted by all physics community. Indeed, in the final session of the 1972 Copenhagen International Conference on Gravitation and Relativity, A. Trautman argued that perhaps many of the difficulties of gravitational theory may be due to the extension of this similarity to all aspects of both fields.

[32] This hypothesis is made here only for simplicity. It is an over-simplification under certain drastic situations, such as very strong gravitational fields.

[33] We are, perhaps, in a situation similar to that experienced by Maxwell, a century ago. His theory described the electromagnetic fields in the interior of substances by means of the same type of fields in vacuum and by characterizing the distortion produced by the matter on the fields, as given by macroscopic quantities: the dielectric constant $\varepsilon_{\mathrm{Max}}$ and the permeability μ_{Max} (the shorthand "Max" represents "Maxwell"). It took many years before Lorentz—who had the atomic theory of matter at his disposal—made Maxwell's theory rigorously understood by averaging properties of microscopic fields on a macroscopic scale.

[34] In the present review, we shall limit ourselves to the sourceless case, i.e., the so-called vacuum gravitational fields. A generalization to include matter is straightforward and presents no difficulties.

respect to ℓ^μ, mixes the terms $E_{\alpha\beta}$, $D_{\alpha\beta}$, $H_{\alpha\beta}$ and $B_{\alpha\beta}$ into the equations. This situation could be thought of as defining a new type of ether. However, unlike the ether of the pre-Einstein epoch, our ether is not a substance, but it is only a preferred frame of observation.

In the remainder of this presentation, we discuss in some detail a very particular situation of these tensors, that is, the case in which they can be reduced to two tensors plus two constants: ε and μ. Then, we show that Einstein's theory is obtained from this for a particular set of values of ε and μ, that is, $\varepsilon = \mu = 1$. It is in this sense that we can call this theory a generalization of Einstein's gravidynamics.

A1 The Q-Field

A1.1 Definitions

Let us define in a four-dimensional Riemannian manifold a fourth-rank tensor $Q_{\alpha\beta\mu\nu}$ described by an observer V^μ in terms of four second-order tensors $E_{\alpha\beta}$, $D_{\alpha\beta}$, $H_{\alpha\beta}$ and $B_{\alpha\beta}$.We set, by analogy with the irreducible decomposition of the Weyl tensor, that

$$Q_{\alpha\beta}{}^{\mu\nu} = V_{[\alpha} D_{\beta]}{}^{[\mu} V^{\nu]} + V_{[\alpha} E_{\beta]}{}^{[\mu} V^{\nu]} + \delta^{[\mu}_{[\alpha} E^{\nu]}_{\beta]} - \eta_{\alpha\beta\rho\sigma} V^\rho B^{\sigma[\mu} V^{\nu]} - \eta^{\mu\nu\rho\sigma} V_\rho H_{\sigma[\alpha} V_{\beta]}. \quad \text{(A1)}$$

The tensors $E_{\alpha\beta}$, $D_{\alpha\beta}$, $B_{\alpha\beta}$ and $H_{\alpha\beta}$, represented below by $X_{\alpha\beta}$, satisfy the following properties:

$$X^\alpha{}_\alpha = 0, \quad \text{(A2a)}$$
$$X^{\alpha\beta} V_\alpha = 0, \quad \text{(A2b)}$$
$$X_{\alpha\beta} = X_{\beta\alpha}. \quad \text{(A2c)}$$

We can write $D_{\alpha\beta}$, $E_{\alpha\beta}$, etc. in terms of $Q_{\alpha\beta\mu\nu}$ and projections on V^μ like, for instance

$$D_{\alpha\beta} = -Q_{\varepsilon\alpha\mu\beta} V^\varepsilon V^\mu$$

and so on.

A1.2 Algebraic Properties

From the definition (A1) of $Q_{\alpha\beta\mu\nu}$, we directly obtain the following properties:

$$Q_{\alpha\beta}{}^{\mu\nu} = -Q_{\beta\alpha}{}^{\mu\nu}, \quad \text{(A3)}$$
$$Q_{\alpha\beta}{}^{\mu\nu} = -Q_{\alpha\beta}{}^{\nu\mu}, \quad \text{(A4)}$$
$$Q^\alpha{}_{\beta\alpha\nu} = E_{\beta\nu} - D_{\beta\nu}, \quad \text{(A5)}$$
$$Q^\alpha{}_\alpha = 0 \quad \text{(A6)}$$

A1.3 Dynamics

By analogy with Einstein's equations in vacuum, we impose on $Q_{\alpha\beta\mu\nu}$ the equation of motion[35]

$$Q^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \quad \text{(A7)}$$

Now, we shall use the above properties to project the system of (A7) parallel and orthogonal to the rest frame of a selected observer with four-velocity ℓ^μ from the whole class of V^μ. We impose that the congruence generated by ℓ^μ satisfy the properties.

$$\ell^\mu \ell_\mu = +1, \quad \text{(A8a)}$$
$$w_{\alpha\beta} = \frac{1}{2} h_{[\alpha}{}^\lambda h_{\beta]}{}^\varepsilon \ell_{\lambda;\varepsilon} = 0, \quad \text{(A8b)}$$
$$\theta_{\alpha\beta} = \frac{1}{2} h_{(\alpha}{}^\lambda h_{\beta)}{}^\varepsilon \ell_{\lambda;\varepsilon} = 0, \quad \text{(A8c)}$$
$$\dot{\ell}^\mu = \ell^\mu{}_{;\nu} \ell^\nu = 0. \quad \text{(A8d)}$$

where $h_{\mu\nu}$ is the projector in the plane orthogonal to ℓ^μ, that is

$$h_{\mu\nu} = g_{\mu\nu} - \ell_\mu \ell_\nu \quad \text{(A9)}$$

So, the congruence generated by ℓ_μ is geodesic, irrotational, nonexpanding and shear-free. The reason for selecting such a particular class of observers will become clear later. Then, (A7) takes the form

$$D_{\alpha\mu;\nu} h^{\mu\nu} h^\alpha{}_\varepsilon = 0, \quad \text{(A10a)}$$
$$\dot{D}_{\alpha\mu} h^\alpha_{(\sigma}\, h^{\ \mu}_{\varepsilon)} + h^\alpha_{(\sigma}\, \eta_{\varepsilon)}{}^{\nu\rho\tau} \ell_\rho H_{\tau\alpha;\nu} = 0, \quad \text{(A10b)}$$
$$B_{\alpha\mu;\nu} h^{\mu\nu} h^\alpha_\varepsilon = 0, \quad \text{(A10c)}$$
$$\dot{B}^{\mu\nu} h_{\mu(\sigma}\, h_{\ \lambda)\nu} - h^{\ \alpha}_{(\sigma}\, \eta_{\ \lambda)}{}^{\nu\rho\tau} \ell_\rho E_{\tau\alpha;\nu} = 0\,, \quad \text{(A10d)}$$

in which a parenthesis means symmetrization.

This set of equations has a striking resemblance with Maxwell's macroscopic equations of electrodynamics. Indeed, we can formally understand the above set as having the form [79, 80]

$$\nabla \cdot \vec{D} = 0\,, \quad \text{(A11a)}$$
$$\dot{\vec{D}} - \nabla \times \vec{H} = 0\,, \quad \text{(A11b)}$$
$$\nabla \cdot \vec{B} = 0\,, \quad \text{(A11c)}$$
$$\dot{\vec{B}} + \nabla \times \vec{E} = 0\,, \quad \text{(A11d)}$$

where the symbol $\rightarrow$ is put over D, E, etc. only to represent its tensorial character; the ∇ operator represents the generalizations of the usual well-known differential operators.

We can therefore understand the reason for selecting the above privileged set of observers, given by the tangential vector ℓ^μ. Only for this class of frames does (A7) take the form (A10a–A10d). Any other observer which is in motion

[35] Indeed, as we shall see, in the case $E_{\alpha\beta} = D_{\alpha\beta}$ and $B_{\alpha\beta} = H_{\alpha\beta}$, $Q_{\alpha\beta\nu\mu}$ can be identified with Weyl's tensor and (10) reduces to Einstein's equation in the vacuum.

with respect to ℓ^μ will mix into the equations of motion of the set of tensors $(E_{\alpha\beta}, B_{\alpha\beta})$ with the set of tensors $(D_{\alpha\beta}, H_{\alpha\beta})$. So, it is in this sense that there is a natural selection of observers, with respect to the equation of motion satisfied by $Q_{\alpha\beta\mu\nu}$.

A1.4 ε and μ States of Tension

A particular class of states of space-time occurs in the case in which there is a linear function relating the tensors $B_{\alpha\beta}$ with $H_{\alpha\beta}$ and $E_{\alpha\beta}$ with $D_{\alpha\beta}$ by the intermediary of two constants, ε and μ.

We set

$$B_{\alpha\lambda} = \mu H_{\alpha\lambda}, \tag{A12a}$$
$$D_{\alpha\lambda} = \varepsilon H_{\alpha\lambda}. \tag{A12b}$$

If we put expressions (A12b) into definition (A1) of $Q_{\alpha\beta\mu\nu}$, a straightforward calculation shows that it is possible to write $Q_{\alpha\beta\mu\nu}$ in terms of the Weyl tensor $W_{\alpha\beta\mu\nu}$ and its "electric" and "magnetic" parts $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$, if we identify the tensor $E_{\alpha\beta}$ with $\mathcal{E}_{\alpha\beta}$ and $H_{\alpha\beta}$ with $\mathcal{H}_{\alpha\beta}$. Then, we can write

$$Q_{\alpha\beta} = W_{\alpha\beta}{}^{\mu\nu} + (\varepsilon - 1)\ell_{[\alpha}\, \mathcal{E}^{[\mu}_{\beta]}\, \ell^{\nu]} + (1-\mu)\eta_{\alpha\beta\rho\sigma}\mathcal{H}^{\sigma[\mu}\, \ell^{\nu]}\ell^\rho\ , \tag{A13}$$

where

$$W_{\alpha\beta}{}^{\mu\nu} = 2\ell_{[\alpha}\, \mathcal{E}_{\beta]}{}^{[\mu}\ell^{\nu]} + \delta^{[\mu}_{[\alpha}\, \mathcal{E}^{\nu]}_{\beta]} - \eta_{\alpha\beta\lambda\sigma}\ell^\lambda\mathcal{H}^{\sigma[\mu}\ell^{\nu]} - \eta^{\mu\nu\rho\sigma}\ell_\rho\mathcal{H}_{\sigma[\alpha}\,\ell_{\beta]} \tag{A14}$$

and, consequently,

$$\mathcal{E}_{\alpha\beta} = -W_{\alpha\mu\beta\nu}\ell^{mu}\ell^\nu, \tag{A15}$$

$$\mathcal{H}_{\alpha\beta} = \frac{1}{2}\,\eta_{\alpha\mu}{}^{\rho\sigma}W_{\rho\sigma\beta\lambda}\ell^\mu\ell^\lambda \tag{A16}$$

The resulting equations of motion (13)–(14) turn into the set:

$$\mathcal{E}_{\alpha\mu||\nu}h^{\mu\nu}h^\alpha{}_\varepsilon = 0, \tag{A17a}$$
$$\varepsilon\dot{\mathcal{E}}_{\alpha\mu}h^\alpha_{(\sigma}\, h^\mu_{\varepsilon)} + h^\alpha_{(\sigma}\, \eta^{\nu\rho\tau}_{\varepsilon)}\ell_\rho\mathcal{H}_{\tau\alpha||\nu} = 0, \tag{A17b}$$
$$\mathcal{H}_{\alpha\mu||\nu}h^{\mu\nu}h^\alpha_\varepsilon =, \tag{A17c}$$
$$\mu\dot{\mathcal{H}}_{\alpha\mu}h^\alpha_{(\sigma}\, h^\mu_{\varepsilon)} - h^\alpha_{(\sigma}\, \eta^{\nu\rho\tau}_{\varepsilon)}\ell_\rho\mathcal{E}_{\tau\alpha||\nu} = 0. \tag{A17d}$$

By the same argument that guided us to (A10a–A10d), we see from the above set that we can identify ε as being the gravitational analogue of the dielectric constant of electrodynamics and μ as being the permeability of space-time.

Now, we recognize in (A17d) Einstein's equations for the free gravitational field for the particular case in which $\varepsilon = \mu = 1$.[36]

[36]This equivalence is only complete if we impose as initial date the set $R_{\mu\nu} = 0$ on a given space-like hypersurface.

So, it seems natural to interpret (A17d) for the general case (ε, μ different from unity) as the equations for the gravitational field on states of space-time that are macroscopically characterized (in the sense discussed in the introduction) by the two constants ε and μ.

A1.5 Conformal Behavior of $Q_{\alpha\beta\mu\nu}$

A conformal transformation of the metric $g_{\alpha\mu}$ is given by the map

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)$$
$$g^{\mu\nu}(x) \longrightarrow \tilde{g}^{\mu\nu}(x) = \Omega^{-2}(x)g^{\mu\nu}(x) \tag{A18}$$

Since we can set $\eta_{\alpha\beta}{}^{\mu\nu}$ as independent of the conformal transformation,

$$\tilde{\eta}_{\alpha\beta}{}^{\mu\nu} = \eta_{\alpha\beta}{}^{\mu\nu}$$

It is then easy to see that the electric and magnetic parts of Weyl tensor remain unchanged,

$$\tilde{\mathcal{E}}_{\mu\nu} = -\widetilde{W}_{\mu\rho\nu\sigma}\tilde{\ell}^\rho\tilde{\ell}^\sigma = \mathcal{E}_{\mu\nu}\ , \tag{A19a}$$
$$\tilde{\mathcal{H}}_{\mu\nu} = -\frac{1}{2}\,\tilde{\eta}_{\mu\alpha}{}^{\rho\sigma}\,\widetilde{W}_{\rho\sigma\nu\lambda}\tilde{\ell}^\alpha\tilde{\ell}^\lambda = \mathcal{H}_{\mu\nu}, \tag{A19b}$$

where we have used conformal transformation of the velocity ℓ^μ as usual,

$$\tilde{\ell}^\mu = \Omega^{-1}\ell^\mu \tag{A20}$$

As a consequence of the above mapping, $Q_{\alpha\beta\mu\nu}$ behaves, under the conformal transformation, as the Weyl tensor,

$$\tilde{Q}_{\alpha\beta\mu\nu}(x) = \Omega^2(x)Q_{\alpha\beta\mu\nu}(x). \tag{A21}$$

A2 Gravitational Energy in an $\varepsilon - \mu$ State of Tension

There have been many discussions, since Einstein's [46, 47] paper, concerning the definition of the energy of a given gravitational field. We do not intend to discuss this subject here but we shall limit ourselves to considering one reasonably successful suggestion by Bel [12] for the form of the energy-momentum tensor of gravitational radiation.

The point of departure [12] comes from the similitude of the equation of motion of gravity and electrodynamics. He defines a fourth-rank tensor $T^{\alpha\beta\mu\nu}$ given in terms of quadratic components of the field (identified with the Riemann tensor) and written in terms of the Weyl tensor $W^{\alpha\beta\mu\nu}$.

Bel's super-energy tensor takes the form:

$$T^{\alpha\beta\mu\nu} = \frac{1}{2}\left\{W^{\alpha\rho\mu\sigma}W^\beta{}_\rho{}^\nu{}_\sigma + W^{*\alpha\rho\mu\sigma}W^{*\beta}{}_\rho{}^\nu{}_\sigma\right\}\ , \tag{A22}$$

where * is the dual operator. Note that the symmetry of the Weyl tensor ($W^*_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu}^* = {}^*W_{\alpha\beta\mu\nu}$) does not hold for $Q_{\alpha\beta\mu\nu}$. This is related to the lack of $Q_{\alpha\beta\mu\nu} \neq Q_{\mu\nu\alpha\beta}$ symmetry. Indeed, we have that

$$Q^{*}_{\alpha\beta}{}^{\mu\nu} = W^{*}_{\alpha\beta}{}^{\mu\nu} + \frac{1}{2}(\varepsilon - 1)\eta_{\alpha\beta\rho\sigma}\ell^{[\rho}\,\mathcal{E}^{\sigma]\,[\mu}\,\ell^{\nu]} + \frac{1}{2}\,(1-\mu)\eta_{\alpha\beta\rho\sigma}\eta^{\rho\sigma\varepsilon\sigma}\ell_{\varepsilon}\mathcal{H}^{[\mu}_{\tau}\,\ell^{\nu]}$$

and

$$Q_{\alpha\beta}{}^{\mu*\nu} = W^{*}_{\alpha\beta}{}^{\mu\nu} + \frac{1}{2}(\varepsilon - 1)\eta^{\mu\nu}{}_{\rho\sigma}\ell^{[\alpha}\,\mathcal{E}^{\beta]\,[\rho}\,\ell^{\sigma]} + \frac{1}{2}\,(1-\mu)\eta^{\mu\nu}{}_{\rho\sigma}\eta_{\alpha\beta\varepsilon\tau}\ell^{\varepsilon}\mathcal{H}^{\tau\,[\rho}\,\ell^{\sigma]}$$

Then we have that

$$Q_{\alpha\beta\mu\nu}^{\;\;*}\ell^{\beta}\ell^{\nu} = W^{*}_{\alpha\beta\mu\nu}\ell^{\beta}\ell^{\nu} = \mathcal{H}_{\alpha\mu}\,,$$
$$Q^{*}_{\alpha\beta\varepsilon\sigma}\ell^{\beta}\ell^{\sigma} = \mu\,\mathcal{H}_{\alpha\varepsilon}\,.$$

This $T^{\alpha\beta\mu\nu}$ tensor has properties that are very similar to the Minkowski energy-momentum tensor ofelectrodynamics. The scalar constructed with $T^{\alpha\beta\mu\nu}$ and the tangent vector ℓ^{ν}, for instance, takes the form

$$U_{(T)} = T^{\alpha\beta\mu\nu}\ell_{\alpha}\ell_{\beta}\ell_{\mu}\ell_{\nu} \tag{A23}$$

and gives the "energy" of the field

$$U_{(T)} = \frac{1}{2}(\mathcal{E}^2 + \mathcal{H}^2), \tag{A24}$$

where

$$\mathcal{E}^2 = \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta}. \tag{A25a}$$
$$\mathcal{H}^2 = \mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta}. \tag{A25b}$$

In the context of the present extended theory, for a space-time in the state $\varepsilon - \mu$ of tension, we are led to modify $T^{\alpha\beta\mu\nu}$ into $\Theta^{\alpha\beta\mu\nu}$ defined in an analogous manner by the equality

$$\Theta^{\alpha\beta\mu\nu} = \frac{1}{2}\left\{Q^{\alpha\rho\mu\sigma}\,W^{\beta}{}_{\rho}{}^{\nu}{}_{\sigma} + Q^{\alpha^*\rho\mu\sigma}\,{}^{*}W^{\beta}{}_{\rho}{}^{\nu}{}_{\sigma}\right\}. \tag{A26}$$

Then, the energy $U_{(\varepsilon,\mu)}$ as viewed by an observer ℓ^{μ} will be given by the relation

$$U_{(\varepsilon,\mu)} = \theta^{\alpha\beta\mu\nu}\ell_{\alpha}\ell_{\beta}\ell_{\mu}\ell_{\nu} = \frac{1}{2}\left(\varepsilon\mathcal{E}^2 + \mu\mathcal{H}^2\right) \tag{A27}$$

in complete analogy with the electrodynamical case in a general medium.

We would like to make an additional remark by presenting two special properties of $\Theta^{\alpha\beta\mu\nu}$

$$\Theta^{\alpha}{}_{\beta\alpha\mu} = \frac{1}{2}(1-\varepsilon)\varepsilon^{\rho\sigma}W_{\beta\rho\mu\sigma}\,, \tag{A28a}$$
$$\Theta = \Theta^{\alpha\mu}{}_{\alpha\mu} = 0\,. \tag{A28b}$$

Property (A28a) states that not all traces of $\Theta^{\alpha\beta\mu\nu}$ are null for a general state of tension of space-time and that the non-null parts of the contracted tensor are independent of the "permeability" μ. The second property (A28b) states that the scalar obtained by taking the trace of $\Theta^{\alpha\beta\mu\nu}$ twice is null, independent from the state of tension of the space-time.

A3 The Velocity of Propagation of Gravitational Disturbances in States of Tension

To determine the velocity of gravitational waves in $\varepsilon - \mu$ states of space-time, let us perturb the set of equations (A10a–A10d). The perturbation will be represented by the map:

$$\mathcal{E}_{\mu\nu} \longrightarrow \mathcal{E}_{\mu\nu} + \delta\mathcal{E}_{\mu\nu}\,, \tag{A29a}$$
$$\mathcal{H}_{\mu\nu} \longrightarrow \mathcal{H}_{\mu\nu} + \delta\mathcal{H}_{\mu\nu} \tag{A29b}$$

where $\delta\mathcal{E}_{\mu\nu}$, $\delta\mathcal{H}_{\mu\nu}$ are null quantities. Then, (A10a–A10d) are transformed into the perturbed set of equations

$$\delta\mathcal{E}_{\alpha}{}^{\beta}{}_{;\beta} \approx 0\,, \tag{A30a}$$
$$\varepsilon\delta\dot{\mathcal{E}}_{\alpha\mu} + \frac{1}{2}\,h^{\lambda}_{(\alpha}\eta_{\mu)}{}^{\rho\sigma\tau}\ell_{\rho}\,\delta\mathcal{H}_{\tau\lambda;\rho} \approx 0\,, \tag{A30b}$$
$$\delta\mathcal{H}_{\alpha}{}^{\beta}{}_{;\beta} \approx 0\,, \tag{A30c}$$
$$\mu\,\delta\dot{\mathcal{H}} - \frac{1}{2}\,h^{\lambda}_{(\alpha}\eta_{\mu)}{}^{\rho\sigma\tau}\mathcal{E}_{\tau\lambda;\rho} \approx 0\,, \tag{A30d}$$

where the covariant derivative is taken in the background—and we limit ourselves to the linear terms of the perturbation.

Now, let us specialize the background to be the flat Minkowski space-time.[37] In this case, the covariant derivatives are the usual derivation and we can use commutative property to write:

$$\varepsilon\delta\ddot{\mathcal{E}}_{\alpha\beta} + \frac{1}{2}\,h^{\lambda}_{(\alpha}\eta^{\rho\sigma\tau}_{\beta)}\ell_{\sigma}\delta\dot{\mathcal{H}}_{\tau\lambda,\rho} \approx 0 \tag{A31}$$

by taking the derivative of (A30b) projected in the privileged direction ℓ^{μ}.

Multiplying (A30d) by the factor

$$\frac{1}{2\mu}\,h^{\nu}_{(\alpha}\eta_{\beta)}{}^{\gamma\sigma\tau}\ell_{\tau}\,\frac{\partial}{\partial x^{\sigma}}\,,$$

we find that

$$\frac{1}{2}h^{\nu}_{(\alpha}\,\eta_{\beta)}{}^{\gamma\sigma\tau}\ell_{\tau}\,\delta\dot{\mathcal{H}}_{\gamma\nu,\sigma} - \frac{1}{4\mu}\,h^{\nu}_{(\alpha}\,\eta_{\beta)}{}^{\sigma\tau\gamma}\,\ell_{\tau}\ell_{\rho}h^{\varepsilon}{}_{(\gamma}\,\eta_{\nu)}{}^{\psi\rho\phi}\delta\varepsilon_{\psi\varepsilon,\psi,\sigma} \approx 0. \tag{A32}$$

Substituting (A31) into (A31) we finally find that

$$\delta\ddot{\mathcal{E}}_{\alpha\mu} - \frac{1}{\varepsilon\mu}\nabla^2\delta\mathcal{E}_{\alpha\mu} = 0\,, \tag{A33}$$

where ∇^2 is the Laplacian operator defined in the three-dimensional space orthogonal to ℓ^{μ}.

In the same way, an analogous wave equation can be obtained for $\mathcal{H}_{\alpha\mu}$. From (A33), we obtain the expected

[37] We are certainly not considering a usual Minkowski space-time. Here, we are considering a more complex structure that takes into account the fluctuations of space-time (as in quantum gravidynamics). These fluctuations are assumed to be represented on an average by the macroscopic quantities ε and μ, as stated in the introduction.

result: the velocity of propagation of gravitational waves in ε, μ states of tension of space-time is equal to $1/\sqrt{\varepsilon\mu}$.

The set of privileged observers that we dealt with here may be enlarged by somehow weakening the defining conditions [see (A8a–A8d)]. This point deserves further investigation.

References

1. M. Abramowitz, I.A. Stegun. *Handbook of Mathematical Functions* (Dover Publications, New York, 1974), p. 504
2. J. Acacio de Barros, N. Pinto-Neto, M.A. Sagioro-Leal, Phys. Lett. A. **241**, 229 (1998)
3. R. Adler, M. Bazin, M. Schiffer. *Introduction to General Relativity, Chap. 6* (McGraw-Hill, New York, 1975)
4. G. Ares de Parga, O. Chavoya, J.L. López Bonilla, J. Math. Phys. **30**, 1294 (1989)
5. E. Ayon-Beato, A. Garcia, Phys. Rev. Lett. **80**, 5056 (1998)
6. E. Ayon-Beato, A. Garcia, Gen. Relat. Gravit. **31**, 629 (1999a)
7. E. Ayon-Beato, A. Garcia, Phys. Lett. B. **464**, 25 (1999b)
8. J. Bardeen. *Proceedings of GR5*, (Tbilisi, U.S.S.R, 1968), p. 174
9. J. Bardeen, Phys. Rev. D. **22**, 1882 (1980)
10. C. Barrabes, V.P. Frolov, Phys. Rev. D. **53**, 3215 (1996)
11. H. Bateman. *The Mathematical Analysis of Electrical and Optical Wave Motion* (Dover, New York, 1955)
12. L. Bel. *Colloques Internationaux du Centre National de la Recherche Scientifique* (Paris, 1962), p. 119
13. V.A. Belinski, I.M. Khalatnikov, E.M. Lifshtz, Usp. Fiz. Nauk. **102**, 463 (1970)
14. V.A. Belinski, I.M. Khalatnikov, E.M. Lifshtz, Adv. Phys. **19**, 525 (1880)
15. I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, G. Veneziano, arXiv:astro-ph.CO/1202.1247v2 (2012)
16. Z. Bialynicka-Birula, I. Bialynicka-Birula, J. Opt. Soc. Am. B. **4**, 10 (1987)
17. J. Bicak, J. Griffiths, Ann. Phys. (N.Y.) **252**, 180 (1996)
18. N.D. Birrel, P.C.W. Davis. *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982)
19. E. Bittencourt, Master's Thesis, CBPF/Rio de Janeiro (2009)
20. H.J. Blome, C. Chicone, F.W. Hehl, B. Mashhoon, arXiv:gr-qc/1002.1425 (2010)
21. P.R. Brady, S. Droz, W. Israel, S.M. Morsink, Class. Quantum Grav. **13**, 2211 (1996)
22. R. Brandenberger, in *Proceedings of the VIII Brazilian School of Cosmology and Gravitation*, ed. by M. Novello (Editions Frontiéres, Singapore, 1996)
23. R. Brandenberger, R. Khan, W.H. Press, Phys. Rev. D. **28**, 1809 (1983)
24. R. Brandenberger, V. Mukhanov, A. Sornborger, Phys. Rev. D. **48**, 1629 (1993)
25. M. Bruni, P.K.S. Dunsby, G.F.R. Ellis, Astrophys. J. **395**, 34 (1992)
26. M. Bruni, S. Matarrese, O. Pantano, Phys. Rev. Lett. **74**, 1916 (1995)
27. A. Cabo, E. Ayon-Beato, arXiv:gr-qc/9704073 (1997)
28. R.R. Caldwell, R. Dare, P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998)
29. M. Campista, Ph. D. Thesis, CBPF/Rio de Janeiro (2010)
30. M. Campista, M. Novello, J.M. Salim, Int. J. Mod. Phys. A. **25**, 3095 (2010)
31. A. Campos, B.L. Hu, Phys. Rev. D. **58**, 125021 (1998)
32. C. Cartier, R. Durrer, E.J. Copeland, Phys. Rev. D. **67**, 103517 (2003)
33. C. Chicone, B. Mashhoon, arXiv:gr-qc/1111.4702 (2008)
34. Y. Choquet-Bruhat, R. Geroch, Comm. Math. Phys. **14**, 329 (1969)
35. Y. Choquet-Bruhat. *General Relativity and 's Equations* (Oxford University Press, New York, 2009)
36. R. Coqueraux, A. Grossmann, Ann. Phys. (N.Y.) **143**, 296 (1982)
37. M. Cvetic, Phys. Rev. Lett. **71**, 815 (1993)
38. M. Dabrowski, J. Stelmach, Astron. J. **92**, 1272 (1986)
39. V.A. De Lorenci, R. Klippert, M. Novello, J.M. Salim, Phys. Rev. D. **65**, 063501 (2002). arXiv:gr-qc/9806076v2 (1998)
40. R. Debever, Cahiers de Physique 168 (1964)
41. P.A.M. Dirac, Can. J. Math. **2**, 129 (1950)
42. G. Dunne, T. Hall, Phys. Rev. D. **58**, 105022 (1998)
43. G. Dunne, Int. J. Mod. Phys. A. **12**, 1143 (1997)
44. P.K.S. Dunsby, B.A.C.C. Bassett, G.F.R. Ellis, Class. Quantum Grav. **14**, 1215 (1997)
45. D. Edwards, Astrophys. Space Sci. **24**, 563 (1973)
46. A. Einstein, Ann. Phys. **354**, 769 (1916)
47. A. Einstein, Rev. Mod. Phys. **20**, 35 (1048)
48. G.F.R. Ellis. *Proceedings of the International School of Physics, Enrico Fermi Course XLVII* (London Academic 104, 1971)
49. G.F.R. Ellis, in *Cargèse Lectures in Physics Vol. VI*, ed. by E. Schatzmann (Gordon and Breach, London, 1971)
50. G.F.R. Ellis, M. Bruni. Phys. Rev. D. **40**, 1804 (1989)
51. G.F.R. Ellis, H. van Elst, in *NATO science series. Ser. C, Math. Phys. Sci.,* vol. 541, ed. by M. Lachièze-Rey (Kluwer Academic, 1999), p. 1
52. J.C. Fabris, J.M. Salim, S.L. Sautu, Mod. Phys. Lett. A. **13**, 953 (1998)
53. M. Fierz, Helv. Phys. Acta. **12**, 379 (1939)
54. R. Garcia-Salcedo, N. Breton, Int. J. Mod. Phys. A. **15**, 4341 (2000)
55. M. Gasperini, G. Marozzi, F. Nugier, G. Veneziano, JCAP. **07**, 008 (2011)
56. M. Gasperini, G. Veneziano, Phys. Rep. **373**, 1 (2003)
57. M. Gasperini, G. Veneziano. arXiv:hep-th/0207130 (2002)
58. M. Giovannini, M. Shaposhnikov, Phys. Rev. D. **57**, 2186 (1998)
59. S.W. Goode, Phys. Rev. D. **39**, 2882 (1989)
60. E. Goulart, Ph.D. Thesis, CBPF/Rio de Janeiro (2007)
61. E. Goulart, M. Novello, Grav. Cosmol. **14**, 321 (2008)
62. I.S. Gradshteyn, I.M. Ryzhik. *Table of Integrals, Series and Products* (Academic, London, 1965)
63. L.P. Grishchuk, Y.V. Sidorov. Phys. Rev. D. **42**, 3413 (1990)
64. S. Hawking, G.F.R. Ellis. *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1977)
65. S. Hawking, Ap. J. **145**, 544 (1966)
66. W. Heisenberg, H. Euler, Z. Phys. **98**, 714 (1936)
67. M. Hindmarsh, A. Everett, Phys. Rev. D. **58**, 103505 (1998)
68. P.A. Hogan, Astrophys. J. **333**, 64 (1988)
69. P.A. Hogan, G.F.R. Ellis, Class. Quantum Grav. **14**, A171 (1997)
70. P.A. Hogan, E.M. O'Shea, Phys. Rev. D. **65**, 124017 (2002)
71. P.A. Hogan, A. Trautman, in *Gravitation and Geometry: Papers in Honour of I. Robinson*, ed. by W. Rindler, A. Trautman (Bibliopolis, Naples, 1987)
72. J.H. Horne, G.T. Horowitz, Nucl. Phys. **B368**, 444 (1992)
73. J.C. Hwang, E.T. Vishniac, Ap. J. **353**, 1 (1990)
74. J. Ibáñez, R.J. van den Hoogen, A.A. Coley, Phys. Rev. D. **51**, 928 (1995)
75. W. Israel, Ann. Phys. **100**, 310 (1976)
76. K. Jedamzik, V. Jatalinić, A.V. Olinto, Phys. Rev. D. **57**, 3264 (1998)
77. B.J.T. Jones, Rev. Mod. Phys. **48**, 107 (1976)

78. P. Jordan, J. Ehlers, R. Sachs, Akad. Wiss. Lt. Meinz Abh. Math. Nat. Kl. **1**, 3 (1961)
79. P. Jordan, J. Ehlers, W. Kundt, Akad. Wiss. Lit. (Mainz) Abhandl. Math.-Nat. Kl. 2. **2**, 21 (1960)
80. P. Jordan, J. Ehlers, W. Kundt, Gen. Rel. Grav. **41**, 2191 (2009)
81. M. Joyce, M. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997)
82. E.G. Kalnins, W. Miller Jr., J. Math. Phys. **32**, 698 (1991)
83. R. Klippert, Ph.D. Thesis, CBPF/Rio de janeiro (1998)
84. R. Klippert, V.A. De Lorenci, M. Novello, J.M. Salim, Phys. Lett. B. **472**, 27 (2000)
85. L. Kofman, A. Linde, A.A. Starobinsky, Phys. Rev. D. **56**, 3258 (1997)
86. C. Lanczos, Rev. Mod. Phys. **34**, 379 (1962)
87. L.D. Landau, E.M. Lifshitz. *Fluid Mechanics* (Pergamon Press, New York, 1982)
88. A. Lichnerowicz, Ann. Mat. Pura Appl. **50**, 1 (1960)
89. E.M. Lifshitz, I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963)
90. M. Marklund, P.K.S. Dunsby, G. Brodin, Phys. Rev. D. **62**, 101501 (2000)
91. M. Mars, M.M. Martin-Prats, J.M.M. Senovila, Clas. Quantum Grav. **13**, L51 (1996)
92. R. Maartens, B.A. Bassett, Class. Quant. Grav. **15**, 705 (1998)
93. B. Mashhoon. Ann. Phys. **17**, 705 (2008). arXiv:gr-qc/0805.2926 (2008)
94. B. Mashhoon, arXiv:gr-qc/1101.3752 (2011)
95. A.L. Matacz, Phys. Rev. D. **49**, 788 (1994)
96. S. Matarrese, O. Pantano, D. Saez, Phys. Rev. D. **47**, 1311 (1993)
97. S. Matarrese, O. Pantano, D. Saez, Phys. Rev. Lett. **72**, 320 (1994)
98. P.G. Miedema, Phys. Rev. D. **50**, 2431 (1994)
99. R. Moessner, M. Trodden. **51**, 2801 (1995). ibid
100. M.C. Motta da Silva, Ph.D. Thesis, CBPF/rio de Janeiro (1998)
101. V. Mukhanov, R. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992)
102. V.F. Mukhanov, H.A. Feldmann, R.H. Brandenberger, Phys. Rep., Vol. 215 (1992)
103. V.F. Mukhanov, S. Winitzki. *Introduction to Quantum Effects in Gravity* (Cambridge University Press, Cambridge, 2007)
104. G.L. Murphy, Phys. Rev. D. **8**, 4231 (1973)
105. H. Mutoh, T. Hirai, K. Maeda, Phys. Rev. D. **55**, 3276 (1997)
106. E.T. Newman, R. Penrose, J. Math. Phys. **3**, 566 (1962)
107. H. Noh, J. Hwang, Phys. Rev. D. **52**, 1970 (1995)
108. M. Novello, Inter. Atomic Energy Agency Ic. **75**(61), 1 (1975)
109. M. Novello, in *Proceedings of the II Brazilian School of Cosmology and Gravitation*, ed. by M. Novello, J. Sansom, Cia, R. de Janeiro (Portuguese, 1980)
110. M. Novello, E. Bittencourt, Gravit. Cosmol. **17**(3), 230 (2011)
111. M. Novello, S.E.P. Bergliaffa, Phys. Rep. **463**, 127 (2008). arXiv:astro-ph/0802.1634v1 (2008)
112. M. Novello, S.L.S. Duque, Phys. A. **168**, 1073 (1990)
113. M. Novello, L.R. de Freitas, Int. J. Mod. Phys. A. **13**, 363 (1998)
114. M. Novello, N. Pinto-Neto, Fortschr. Phys. **40**, 173 (1992a)
115. M. Novello, N. Pinto-Neto, Fortschr. Phys. **40**, 195 (1992b)
116. M. Novello, L. Oliveira, J.M. Salim, E. Elbaz, Int. J. Mod. Phys. D. **1**, 641 (1993)
117. M. Novello, J.M. Salim, Phys. Rev. D. **20**, 377 (1979)
118. M. Novello, J.M. Salim, Fund. Cosmic Phys. **8**, 201 (1983)
119. M. Novello, J.M. Salim, CBPF-NF-021/85, Rio de Janeiro (1985)
120. M. Novello, J.M. Salim, in *The Tenth Marcel Grossmann Meeting*, ed. by M. Novello, S. E. P. Bergliaffa, R. Ruffini (World Scientific, Cingapura, 2006)
121. M. Novello, J.M. Salim, H. Heintzmann, in *J. Leite Lopes Festschrift*, ed. by S. Joffily, (1988)
122. M. Novello, J.M. Salim, M.C. Motta da Silva, S.E. Jorás, R. Klippert, Phys. Rev. D. **51**, 450 (1995a)
123. M. Novello, J.M. Salim, M.C. Motta da Silva, S.E. Jorás, R. Klippert, Phys. Rev. D. **52**, 730 (1995b)
124. M. Novello, J.M. Salim, M.C. Motta da Silva, R. Klippert, Phys. Rev. D. **54**, 2578 (1996)
125. M. Novello, J.M. Salim, M.C. Motta da Silva, R. Klippert, Phys. Rev. D. **61**, 124025 (2000)
126. M. Novello, A.L. Velloso, Gen. Rel. Grav. **19**, 1251 (1987)
127. H.P. De Oliveira, J.M. Salim, S.L. Sautu, Class. Quantum Grav. **14**, 2833 (1997)
128. D.W. Olson, Phys. Rev. D. **14**, 327 (1976)
129. A. Papapetrou. *Lectures on General Relativity* (D. Reidel Publishing Company, Dordrecht, 1974)
130. P.J.E. Peebles. *The Large-Scale Structure of the Universe* (Princeton University Press, Princeton, 1980)
131. P. Peter, N. Pinto-Neto, Phys. Rev. D. **65**, 023513 (2001)
132. F.A.E. Pirani, Phys. Rev. **105**, 1089 (1957)
133. E.M. O'Shea, Phys. Rev. D. **69**, 064038 (2004)
134. E.M. O'Shea. arXiv:gr-qc/0401106 (2004)
135. T. Regge, J.A. Wheeler, Phys. Rev. **108**, 1063 (1957)
136. H.P. Robertson, Rev. Mod. Phys. **5**, 62 (1933)
137. I. Robinson, J. Math. Phys. **2**, 290 (1961)
138. I. Robinson, A. Trautman, J. Math. Phys. **24**, 1425 (1983)
139. M. P. Ryanm Jr. *Lecture on VI Brazilian School of Cosmology and Gravitation* (1991)
140. B.L. Schumaker, Phys. Rep. **135**, 317 (1986)
141. J.M. Salim, Ph.D. thesis, CBPF/Rio de Janeiro (1982)
142. J.M. Salim, H.P. de Olivera, Acta Phys. Pol. B. **19**, 649 (1988)
143. J.M. Salim, S.L. Sautu, Class. Quantum Grav. **13**, 353 (1996)
144. A. Saa, E. Gunzig, L. Brenig, V. Faraoni, T.M. Rocha Filho, A. Figueiredo. arXiv:gr-qc/0012105 (2000)
145. A. Sagnotti, B. Zwiebach, Phys. Rev. D. **24**, 305 (1981)
146. W. de Sitter, Proc. K. Ned. Akad. Wet. **19**, 1217 (1917)
147. J.M. Stewart, Class. Quantum Grav. **7**, 1169 (1990)
148. J.M. Stewart, M. Walker, Proc. R. Soc. Lond. A. **341**, 49 (1974)
149. K. Subramanian, J.D. Barrow. ibid. **58**, 883502 (1998)
150. T. Tajima, S. Cable, K. Shibata, R.M. Kulsrud, Astrophys. J. **390**, 309 (1992)
151. C. Thompson, O. Blaes, Phys. Rev. D. **57**, 3219 (1998)
152. R.C. Tolman, P. Ehrenfest, Phys. Rev. **36**, 1791 (1930)
153. A. Trautman. *Recent Developments in General Relativity* (PWN, Warsaw, 1962)
154. C.G. Tsagas, Class. Quant. Grav. **22**, 393 (2005)
155. C.G. Tsagas, A. Challinor, R. Maartens, Phys. Rep. **465**, 61 (2008)
156. A.A. Tseytlin, Phys. Lett. B. **363**, 223 (1995)
157. G. Veneziano. arXiv:hep-th/0002094 (2000)
158. C.V. Vishveshwara, Phys. Rev. D. **1**, 2870 (1970)
159. J. Wainwright, B.J. Marshman, Phys. Lett. **72A**, 275 (1979)
160. R.M. Wald. *General relativity* (Univ. Chicago Press, Chicago, 1984)
161. H. Weyl. *Space, time and matter* (Dover Publications INC., USA, 1922)
162. D.L. Wiltshire, Class. Quantum Grav. **28**, 164006 (2011)
163. D.L. Wiltshire, P.R. Smale, T. Mattsson, R. Watkins. arXiv:astro-ph.CO/1201.5371v1 (2012)
164. V.D. Zakharov. *Gravitational Waves in Einstein's Theory* (Halsted Press, Jerusalem, 1973)