PHYSICAL REVIEW D **86,** 124024 (2012)

Gordon metric revisited

M. Novello* and E. Bittencourt†

Instituto de Cosmologia Relatividade Astrofisica ICRA—CBPF Rua Doutor Xavier Sigaud, 150, CEP 22290-180 Rio de Janeiro, Brazil

(Received 10 October 2012; published 7 December 2012)

We show that the Gordon metric belongs to a larger class of geometries, which are responsible to describe the paths of accelerated bodies in moving dielectrics as geodesics in a metric $\hat{q}_{\mu\nu}$ different from the background one. This map depends only on the background metric and on the motion of the bodies under consideration. As a consequence, this method describes a more general property that concerns the elimination of any kind of force acting on bodies by a suitable change of the substratum metric.

DOI: 10.1103/PhysRevD.86.124024 PACS numbers: 04.20.−q

I. INTRODUCTION

In 1923 Gordon [1] made a seminal suggestion to treat the propagation of electromagnetic waves in a moving dielectric, modifying the metric structure of the background. He showed that the electromagnetic waves propagate as geodesics not in the background geometry $\eta_{\mu\nu}$ but instead in the effective metric

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} + (\epsilon\mu - 1)v^{\mu}v^{\nu}, \tag{1}$$

where ϵ and μ are constant parameters that characterize the dielectric and v^{μ} is the four-velocity of the material under consideration (which is not necessarily constant). Later, it was recognized that this interpretation could be used to describe nonlinear structures even when ϵ and μ depend on the intensity of the electromagnetic field [2] or more complicated functions of the field strengths [3]. In all these cases, the causal cone, which is associated to the effective metric, does not coincide with the null-cone of the theory. The origin of this modification is due to the presence of a moving dielectric, which changes the paths of the electromagnetic waves inside this medium.

We then face the question: could such particular description of the electromagnetic waves in moving dielectrics be generalized for other cases, in which accelerated paths due to other kind of forces would be described as geodesic motions in an associated metric? We shall see that the answer is affirmative and this kinematical map depends only upon the acceleration of the body and the background metric.

This method allows us to geometrize any force in the sense that an arbitrary accelerated body in a given metric substratum $g_{\mu\nu}$ is equivalently described as geodesic motion in an effective geometry $\hat{q}_{\mu\nu}$. We start by analyzing the generalization of the Gordon metric concerning the propagation of electromagnetic waves inside arbitrary dielectric media. This procedure mimics the trajectory which led to the geometrization of the gravitational field as it was done by general relativity (GR). This means to describe the effects of acceleration of a particle on a gravitational field by a change of the space-time metric, according to Einstein's approach. We compare this effective geometry to the metric in the post-Newtonian approximation in order to see if it is possible to reproduce some results of GR. Such procedure is exactly what happens in the analog models of gravitation that deals with systems kinematically equivalence, but dynamically distinct.

II. GEOMETRIZING ACCELERATED PATHS

Consider a vector field[1] u_{μ} with norm $N \equiv u^{\alpha}u_{\alpha}$ in a given background $g_{\mu\nu}$. The acceleration of u_{μ} is given by

$$a_{\mu} = u_{\mu;\nu}u^{\nu}.$$

Now let us construct an associated metric tensor $\hat{q}_{\mu\nu}$, in which the vector u_{μ} satisfies the equation

$$u_{\mu\|\nu}\hat{u}^{\nu} = f(p)u_{\mu}, \tag{2}$$

where $\|$ means covariant derivative with respect to $\hat{q}_{\mu\nu}$ and $f(p)$ is an arbitrary function of the parameter p along the curve. The contravariant components of the vector field are defined by $\hat{u}^{\mu} \equiv \hat{q}^{\mu\nu}u_{\nu}$ and, consequently, the norm defined in $\hat{q}_{\mu\nu}$ is given by $\hat{N} \equiv \hat{q}^{\mu\nu}u_{\mu}u_{\nu}$. Whenever u_{μ} is either a gradient or a normalized vector field, $f(p)$ can be set equal to zero without loss of generality.

Developing Eq. (2), we obtain

$$\frac{1}{2}\hat{N}_{,\mu} + u_{[\mu,\nu]}\hat{u}^{\nu} = f(p)u_{\mu}, \tag{3}$$

where [ ] means skew-symmetrization. Choosing $\hat{q}_{\mu\nu}$ such that this equation is verified, then the accelerated path of u_{α} in $g_{\mu\nu}$ becomes a geodesic motion in the associated $\hat{q}_{\mu\nu}$. Note that this is independent of its functional form. We will show that the Gordon metric is a particular example of this procedure and that there is a class of metrics

*novello@cbpf.br
†eduhsb@cbpf.br

[1] The natural velocity field we use to develop this section is a given 1-form field u_{μ}. In the Gordon approach, the wave vector k_{μ} is a gradient and, therefore, an exact 1-form.

which play the same role in such "unforced-motion" process exhibiting different geometrical properties for each element of the class.

III. LIGHT PATHS ON MOVING DIELECTRIC: GORDON APPROACH

Let us define two skew-symmetric tensors $F_{\mu\nu}$ and $P_{\mu\nu}$ representing the electromagnetic field inside the material medium. These tensors are expressed in terms of the field strengths E^μ and H^ν and field excitations D^μ and B^μ as follows

$$F_{\mu\nu} \equiv E_\mu v_\nu - E_\nu v_\mu + \eta_{\mu\nu}{}^{\alpha\beta} v_\alpha B_\beta,$$
$$P_{\mu\nu} \equiv D_\mu v_\nu - D_\nu v_\mu + \eta_{\mu\nu}{}^{\alpha\beta} v_\alpha H_\beta,$$

where v^μ is a given four-vector comoving with the dielectric and $\eta_{\mu\nu}{}^{\alpha\beta}$ is the Levi-Civita tensor. We assume that the electromagnetic properties of the medium are characterized by the constitutive relations

$$D_\alpha = \epsilon_\alpha{}^\nu(E, H) E_\nu, \qquad B_\alpha = \mu_\alpha{}^\nu(E, H) H_\nu,$$

where $\epsilon_\alpha{}^\nu(E, H)$ and $\mu_\alpha{}^\nu(E, H)$ are arbitrary tensors. Consider Maxwell equations on dielectric media [4] with permittivity ϵ and permeability μ that characterize the dielectric:

$$P^{\mu\nu}{}_{;\nu} = 0, \qquad {}^*F^{\mu\nu}{}_{;\nu} = 0. \tag{4}$$

From now on, we take the background metric as flat Minkowski space-time and assume that $\mu \equiv \mu_0$ is a constant and $\epsilon = \epsilon(E)$, where $E \equiv \sqrt{-E_\alpha E^\alpha}$ and E^α is the electric field. It is straightforward to generalize these equations to arbitrary curved space-time. Indeed, suppose an observer with velocity v^μ comoving with the dielectric and such that $v^\mu{}_{;\nu} = 0$. Then, Eqs. (4) written in terms of the displacement vectors D^μ and B^μ become

$$\begin{aligned} D^\mu{}_{;\nu} v^\nu - D^\nu{}_{;\nu} v^\mu + \eta^{\mu\nu\alpha\beta} v_\alpha H_{\beta;\nu} &= 0, \\ B^\mu{}_{;\nu} v^\nu - B^\nu{}_{;\nu} v^\mu - \eta^{\mu\nu\alpha\beta} v_\alpha E_{\beta;\nu} &= 0. \end{aligned} \tag{5}$$

The projection with respect to v^μ yields the four independent nonlinear equations of motion describing the electromagnetic field inside the dielectric medium:

$$\begin{aligned} &\epsilon E^\alpha{}_{;\alpha} - \frac{\epsilon' E^\alpha E^\beta}{E} E_{\alpha;\beta} = 0, \qquad \mu_0 H^\alpha{}_{;\alpha} = 0, \\ &\epsilon \dot{E}^\lambda - \frac{\epsilon' E^\lambda v^\alpha E^\mu}{E} E_{\mu;\alpha} + \eta^{\lambda\beta\rho\sigma} v_\rho H_{\sigma;\beta} = 0, \\ &\mu_0 \dot{H}^\lambda - \eta^{\lambda\beta\rho\sigma} v_\rho E_{\sigma;\beta} = 0. \end{aligned} \tag{6}$$

We define the unitary vector l^μ by setting $E^\mu \equiv E l^\mu$, where l^μ satisfies $l_\alpha l^\alpha = -1$.

We use Hadamard conditions [5] to obtain the propagation waves through the characteristics surface Σ (for details, see the Appendix). The symbol $[X]_\Sigma$ represents the discontinuity of X through this surface. Then, the discontinuities of Eqs. (6) become

$$[E_{\mu,\lambda}]_\Sigma = e_\mu k_\lambda, \qquad [H_{\mu,\lambda}]_\Sigma = h_\mu k_\lambda, \tag{7}$$

where $e_\mu(x)$ and $h_\mu(x)$ are the amplitudes of the discontinuities and $k_\mu \equiv \partial_\mu \Sigma$ is the wave vector. Thus, it follows that

$$\begin{aligned} &\epsilon k^\alpha e_\alpha - \frac{\epsilon'}{E} E^\alpha e_\alpha E^\beta k_\beta = 0, \qquad \mu_0 h^\alpha k_\alpha = 0, \\ &\epsilon k^\alpha v_\alpha e^\mu - \frac{\epsilon'}{E} E^\lambda e_\lambda v^\alpha k_\alpha E^\mu + \eta^{\mu\nu\alpha\beta} k_\nu v_\alpha h_\beta = 0, \\ &\mu_0 k_\alpha v^\alpha h^\lambda - \eta^{\lambda\beta\rho\sigma} k_\beta v_\rho e_\sigma = 0, \end{aligned} \tag{8}$$

where ϵ' is the derivative of ϵ with respect to E. Combining these equations we obtain the following intermediary relation

$$\begin{aligned} &\frac{e^\mu}{\mu_0 k_\alpha v^\alpha}[k^\nu k_\nu - (k^\nu v_\nu)^2] - \frac{k^\beta e_\beta}{\mu_0 k_\alpha v^\alpha} k^\mu \\ &\quad + \epsilon k^\alpha v_\alpha e^\mu - \frac{\epsilon'}{E} E^\lambda e_\lambda v^\alpha k_\alpha E^\mu = 0, \end{aligned} \tag{9}$$

which multiplying by E_μ yields the dispersion relation

$$\left(\eta^{\mu\nu} + (\mu_0\epsilon - 1 + \mu_0\epsilon' E) v^\mu v^\nu - \frac{\epsilon'}{\epsilon E} E^\mu E^\nu\right) k_\mu k_\nu = 0. \tag{10}$$

We see that the envelop of discontinuity propagates differently from Minkowski light-cone of the linear Maxwell theory. In this case, the causal structure is given by an effective Riemannian geometry[2] $\hat{g}^{\mu\nu}$. From this point of view, k_μ is nulllike in $\hat{g}^{\mu\nu}$, namely,

$$\hat{g}^{\mu\nu} k_\mu k_\nu = 0. \tag{11}$$

The expression of the effective geometry is given by

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} + (\mu_0\epsilon - 1 + \mu_0\epsilon' E) v^\mu v^\nu - \frac{\epsilon' E}{\epsilon} l^\mu l^\nu. \tag{12}$$

A simple calculation show that its inverse is

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} - \left(1 - \frac{1}{\mu_0\epsilon(1+\xi)}\right) v_\mu v_\nu + \frac{\xi}{1+\xi} l_\mu l_\nu, \tag{13}$$

where

$$\xi \equiv \frac{\epsilon' E}{\epsilon}.$$

In particular, when ϵ is a constant, this formula reduces to Gordon's pioneer work, in which was shown that the waves propagate as geodesics not in the background geometry $\eta_{\mu\nu}$ but instead in the effective metric

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} + (\epsilon\mu_0 - 1) v^\mu v^\nu, \tag{14}$$

[2]Mathematically, the metric tensor is a covariant tensor of rank 2. However, in this paper, we sometimes shall call "metric" a contravariant tensor of rank 2. In particular, that is the way the Gordon metric appears naturally.

which depends only on the dielectric properties μ_0, ϵ and v^μ. The magnitude N of the wave vector in Minkowski space-time (written in terms of dielectric properties) is determined by the Gordon relation

$$\hat{g}^{\mu\nu}k_\mu k_\nu = (\eta^{\mu\nu} + (\epsilon\mu_0 - 1)v^\mu v^\nu)k_\mu k_\nu = 0, \longrightarrow N = (1 - \mu_0\epsilon)(k.v)^2, \quad (15)$$

where $k.v \equiv k_\alpha v^\alpha$.

The analysis of the wave propagation in material media and the study of effective geometry are particularly interesting in the investigation of analog model [2,6] for the understanding of kinematical properties at very small scale of astrophysical objects (see details in Refs. [7,8]). We quote Hawking radiation [9] and Unruh's work on experimental black hole evaporation [10] which are systematically studied and the modeling of specific dielectric media is developed in order to eventually detect these tiny effects. A more complete discussion on this topic was given in Refs. [11,12] and references therein. Here we shall point the similarity between these geometries in a later section.

IV. BINOMIAL METRICS

In recent years, an intense activity concerning features of Riemannian geometries similar to those described by the Gordon approach has been done [6]. In particular, that allows a binomial form for both the metric and its inverse. That is, its covariant and the corresponding contravariant expressions are

$$\hat{q}_{\mu\nu} = A\eta_{\mu\nu} + B\Phi_{\mu\nu}, \quad (16)$$

and

$$\hat{q}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta\Phi^{\mu\nu}. \quad (17)$$

This form of the metric requires that the tensor $\Phi^{\mu\nu}$ must satisfy the condition

$$\Phi_{\mu\nu}\Phi^{\nu\lambda} = m\delta^\lambda_\mu + n\Phi^\lambda_\mu. \quad (18)$$

Such feature allows us to write the inverse metric similarly to the binomial form of the metric, avoiding difficulties with infinite series.[3] Two remarkable examples of this property are the scalar field (in which $\Phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi$) and the electromagnetic field (in which $\Phi_{\mu\nu} = F_\mu{}^\alpha F_{\alpha\nu}$).

[3]This is the case of GR as a field theory formulation. The exact expression for metric tensor is set

$$g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}.$$

A consequence is that its inverse, the covariant tensor $g_{\mu\nu}$ is an infinite series:

$$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + h_{\mu\alpha}h^\alpha{}_\nu - h_{\mu\alpha}h^\alpha{}_\beta h^\beta{}_\nu + \cdots$$

This formulation was introduced by Feynman, Gupta and others (cf. Ref. [13]).

A. Special case

In this section, we limit our analysis to the simplest form by setting $\Phi^{\mu\nu} = u^\mu u^\nu$. In this case, the coefficients of the covariant and contravariant forms are related by

$$A = \frac{1}{\alpha}, \qquad B = -\frac{\beta}{\alpha(\alpha+\beta)},$$

where we set $u_\mu u_\nu \eta^{\mu\nu} = 1$ and write the metric in the form

$$\hat{q}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta u^\mu u^\nu.$$

The associated covariant derivative is defined by

$$u^\alpha{}_{\|\mu} = u^\alpha{}_{,\mu} + \hat{\Gamma}^\alpha_{\mu\nu}u^\nu,$$

where the corresponding Christoffel symbol is constructed using $\hat{q}^{\mu\nu}$. The description of an accelerated curve[4] in the flat space-time as a geodesics in the metric $\hat{q}_{\mu\nu}$ is possible if the following condition is satisfied

$$(u_{\mu,\nu} - \hat{\Gamma}^\epsilon_{\mu\nu}u_\epsilon)\hat{u}^\nu = 0, \quad (19)$$

where we have used the metric $\hat{q}^{\mu\nu}$ to write $\hat{u}^\mu \equiv \hat{q}^{\mu\nu}u_\nu = (\alpha+\beta)u^\mu$. Therefore,

$$(u_{\mu,\nu} - \hat{\Gamma}^\epsilon_{\mu\nu}u_\epsilon)u^\nu = 0. \quad (20)$$

In order to preserve the norm $\hat{u}^\mu\hat{u}_\mu$ along the curve we assume $\beta_{,\mu}u^\mu = 0$, without loss of generality (it corresponds to a simple re-parametrization along the curves). Once the acceleration in the background is defined by $a_\mu = u_{\mu,\nu}u^\nu$, the condition of geodetic motion in the $\hat{q}_{\mu\nu}$-geometry takes the form

$$a_\mu = \hat{\Gamma}^\epsilon_{\mu\nu}u_\epsilon u^\nu. \quad (21)$$

The Christoffel symbol reduces to

$$\hat{\Gamma}^\epsilon_{\mu\nu}u_\epsilon u^\nu = \frac{\alpha+\beta}{2}u^\alpha u^\nu \hat{q}_{\alpha\nu,\mu}. \quad (22)$$

Using the expression of $\hat{q}_{\alpha\beta}$ in Eq. (22) and substituting the result into the condition (21), it follows that

$$a_\mu = -\frac{1}{2}\frac{\partial_\mu(\alpha+\beta)}{(\alpha+\beta)}.$$

It means that the acceleration vector a_μ must be a gradient of a function Ψ, i.e.,

$$a_\mu \equiv \partial_\mu\Psi. \quad (23)$$

Thus, the expression of the coefficients α and β of the metric $\hat{q}^{\mu\nu}$ are given in terms of the potential Ψ of the acceleration by

$$\alpha + \beta = e^{-2\Psi}. \quad (24)$$

[4]Note that we are dealing with a collection of paths Γ that is usually called a congruence of curves. It is understood that each element of this collection concern particles that have the same characteristics. For instance, if the acceleration is due to an electromagnetic field, all particles of Γ must have the same relation between its charge and mass, to wit a bunch of electrons.

This simple example gives a very useful formula, which exhibits the connection between geometrical and mechanical quantities. Later in this paper, we shall analyze the case in which Ψ represents the gravitational potential.

B. Polynomial metrics

Gordon approach depends explicitly on the velocity v^α of the dielectric. Nevertheless such form of introducing an effective metric is not unique. Indeed it seems reasonable to present another metric $\hat{q}_{\mu\nu}$ that describes the same results obtained by Gordon and besides reduces the dependence on the four-velocity v^α. From practical reasons, it might be useful to weaken this constraint of Gordon approach, once it is easier to determine the shape of the electromagnetic wave packet in the laboratory than constructing nonlinear dielectric media with arbitrary tensorial parameters $\epsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$—despite of the great advances in this research area recently [14,15].

Let us now show that exists a class of geometries which play the same role as the Gordon metric depending only on the angle $k_\alpha v^\alpha$ between the wave vector k_α and the dielectric four-vector v^α. This is achieved by generalizations of the last section. Let us list some examples:

Case A: the metric $\hat{q}^{\mu\nu}$ is given by

$$\hat{q}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta k^\mu k^\nu,$$

and its inverse is

$$\hat{q}_{\mu\nu} = \frac{1}{\alpha}\eta_{\mu\nu} - \frac{\beta}{\alpha(\alpha + \beta N)}k_\mu k_\nu.$$

Once the wave vector k_μ is a gradient of a given hypersurface Σ, then we have

$$k_{[\mu,\nu]} = 0.$$

Substituting this result in Eq. (3), it follows that k_μ must satisfy

$$\hat{N}_{(q),\mu} = 0 \longrightarrow \hat{N}_{(q)} \equiv \text{const},$$

where we define $\hat{N}_{(q)} \equiv \hat{q}^{\mu\nu}k_\mu k_\nu$. That is, in order to follow a geodesic motion in $\hat{q}_{\mu\nu}$, k_μ must have constant norm in the metric $\hat{q}_{\mu\nu}$. The explicit expression for this constraint is

$$\hat{N}_{(q)} = (\alpha + \beta N)N \equiv 1. \tag{25}$$

Note that this approach transforms the nonnormalized wave vector k_μ in Minkowski background in a normalized timelike vector in $\hat{q}_{\mu\nu}$. It does not violate Lorentz invariance, because everything happens inside the dielectric. We note that it is not possible to fix $\hat{N}$ equal to zero, otherwise the metric is ill-defined. Therefore, k_μ is not a nulllike vector in the $\hat{Q}$-metric. Another feature is that the magnitude of the dielectric four-vector

$$\hat{q}^{\mu\nu}v_\mu v_\nu = \alpha + \beta(k.v)^2,$$

is not necessarily positive definite allowing observers with velocity great than speed of light inside the medium.[5] For instance, if we set

$$\alpha + \beta(k.v)^2 = 0,$$

then, using Eq. (15), we obtain

$$\beta = \frac{(\mu_0\epsilon - 1)\alpha}{N}.$$

Substituting this result in Eq. (25), yields

$$\alpha = \frac{1}{\mu_0\epsilon N}.$$

Therefore, the metric $\hat{q}^{\mu\nu}$ with these values of α and β produces the following outcome: the wave vector k_μ becomes a normalized and timelike vector, while the dielectric velocity v^μ, which was a timelike vector in the Minkowski background, becomes a null geodesic in $\hat{q}_{\mu\nu}$. Therefore, the causal structure is no more determined by k_μ.

Remark that the metric $\hat{q}^{\mu\nu}$ presented in the precedent sections is not unique. We can enlarge the set of metrics that have the same properties showed above adding other terms to $\hat{q}^{\mu\nu}$ provided the condition (18) is valid. To exemplify these cases we consider:

Case B: the polynomial metric is given by

$$\hat{m}^{\mu\nu} = \eta^{\mu\nu} + \beta k^\mu k^\nu + \delta a^\mu a^\nu.$$

It is straightforward to show that its inverse has an extra term

$$\hat{m}_{\mu\nu} = \eta_{\mu\nu} + Bk_\mu k_\nu + \Delta a_\mu a_\nu + \Lambda a_{(\mu}k_{\nu)},$$

where () means symmetrization. The coefficients of the inverse metric are

$$B = -\frac{\beta(1 - \delta a^2)}{X},$$

$$\Delta = -\frac{\delta(1 + \beta N)}{X},$$

and

$$\Lambda = \frac{\beta\delta\dot{N}}{2X}.$$

Here, we defined $a^2 \equiv -a^\alpha a_\alpha$, $\dot{N} \equiv N_{,\mu}k^\mu$ and

$$X = 1 - \delta a^2 + \beta N - \beta\delta\left(\frac{\dot{N}^2}{4} + Na^2\right).$$

[5] In the laboratory, the angle between these two vectors is easier to manipulate than the dielectric velocity field only. We expect that this fact could be of reasonable utility in the research of analog models.

The appearance of an extra term also happens with the inverse metric when we consider instead of $a^\mu a^\nu$ a term of the form $a^{(\mu}k^{\nu)}$. In both cases an extra term is necessary breaking the polynomial symmetry between the metric and its inverse. Nevertheless we will present the calculations for this case focusing only on the metric containing the term $a^\mu a^\nu$ and indicating that the results are very similar when the other term is considered separately.

The geodetic motion condition for the wave vector leads to

$$\hat{N}_{(m)} = (1 + \beta N)N + \frac{\delta}{4}\dot{N}^2 = 0.$$

Note that this approach permits a null geodesic motion for the wave vector k_μ. This is the simplest case in which we regain the main Gordon result (k_μ as a null geodesic). The sign of the norm of v^μ is undetermined and may be chosen equal to zero, as we saw in the previous case.

Case C: the most general case involving first order derivatives of k_μ occurs when the metric is expressed in the form[6]

$$\hat{n}^{\mu\nu} = \alpha\eta^{\mu\nu} + \beta k^\mu k^\nu + \delta a^\mu a^\nu + \lambda a^{(\mu}k^{\nu)}$$

and its inverse is

$$\hat{n}_{\mu\nu} = \frac{1}{\alpha}\eta_{\mu\nu} + Bk_\mu k_\nu + \Delta a_\mu a_\nu + \Lambda a_{(\mu}k_{\nu)}.$$

The covariant metric coefficients are given by

$$B = -\frac{\beta(\alpha - \delta a^2) + \lambda a^2}{Z},$$

$$\Delta = -\frac{\delta(\alpha + \beta N) - \lambda^2 N}{Z},$$

and

$$\Lambda = -\frac{\lambda(2\alpha + \dot{N}\lambda) - 2\beta\delta\dot{N}}{2Z},$$

where

$$Z = \alpha\bigg[\alpha^2 - \alpha\delta a^2 + \alpha\beta N + \alpha\dot{N}\lambda - (\beta\delta - \lambda^2)\bigg(\frac{\dot{N}^2}{4} + Na^2\bigg)\bigg].$$

Once it involves more degrees of freedom, we can regain all outcomes presented before, but with different algebraic relations. In particular, the magnitude of the wave vector in $\hat{n}_{\mu\nu}$ is set

$$\hat{N}_{(n)} = (\alpha + \beta N + \lambda\dot{N})N + \frac{\delta}{4}\dot{N}^2.$$

[6]If we use higher derivatives of k_μ greater than that which appears in the dynamics, then the metric tensor $\hat{q}_{\mu\nu}$ is ill-defined.

Remark that the metric and its inverse have the same number of polynomial terms as required from the beginning. It did not happen in the case B where an extra term was necessary in the inverse metric expression. Following this reasoning, in the next section we shall use only the cases A and C which satisfy the conditions (16) and (17). Moreover, as an example, we will set the potential Ψ as being the Newtonian potential.

C. Application: identifying Ψ with the gravitational potential

In the weak field limit the description of Newton's gravity can be formulated in terms of a geometric representation of the gravitational field. It can be done making use of effective potentials, which correspond to the well-known parameterized post-Newtonian approximation (PPN) [16]. In this section, we compare some PPN results—particularly, those concerning general relativity (GR) predictions, which improve the Newtonian theory of gravity in the solar system—with some $\hat{q}$—metric given by the free-falling (geodesic) condition. We stress that $\hat{q}_{\mu\nu}$ just mimic the geodesic motion characteristic of the solutions of GR. Note that there is no dynamical equation for $\hat{q}_{\mu\nu}$ and we are not proposing such theory. It is purely a kinematical analogy.

This section shows that the analysis through geodesic paths are much more general than GR, once it is nothing but a choice of the metric. According to Poincaré's ideas upon geometrical descriptions of the world [17]: *non-Euclidean geometry is as legitimate as our ordinary Euclidean space; the enunciation of the Physics in this modified geometry would become more complicated, but it still would be possible.* In other words, it is possible to choose the metric of the space-time, such that an accelerated motion in a given geometry can be described as geodesic in another one.

For convenience, consider Minkowski space-time in spherical coordinates

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2,$$

and an observer field $u_\mu = (1, f(r), 0, 0)$, which is a gradient $u_\mu \equiv \partial_\mu \Sigma$, where $\Sigma = t \pm F(r)$.[7] This vector has a nonnull acceleration given by

$$a_\mu = u_{\mu;\nu}u^\nu = \frac{1}{2}N_{,\mu} = (0, -ff', 0, 0),$$

where prime $'$ means derivative with respect to radial coordinate r and $N \equiv u_\mu u_\nu \eta^{\mu\nu}$.

We choose the scalar function Ψ, which characterizes the acceleration $a_\mu \equiv \partial_\mu \Psi$, as identified with the Newtonian potential. Then, it follows a relation between N and Ψ given by

[7]This situation can perfectly be adapted to describe wave vectors in material media.

$$N = 1 + 2\Psi = 1 - \frac{r_H}{r},$$

where $r_H \equiv 2M$ and M is the mass source of the gravitational field ($G = c = 1$). To go further in the calculations, we basically split the analysis in two distinct cases. One of them gets immediately a wrong linear regime compared to GR, while the other case, which is separated in two subcases, can give the expected weak field regime.

Case I: consider the $\hat{q}$-metric as follows

$$\hat{q}^{\mu\nu} = \eta^{\mu\nu} + \beta u^\mu u^\nu. \quad (26)$$

The inverse metric is

$$\hat{q}_{\mu\nu} = \eta_{\mu\nu} - \frac{\beta}{1+\beta N} u_\mu u_\nu. \quad (27)$$

The condition for u_μ to follow a geodesic motion in this metric is provided by

$$u_{\mu\|\nu}\hat{u}^\nu = \frac{1}{2}\hat{N}_{(q),\mu} = 0, \quad (28)$$

where $\hat{u}^\mu \equiv \hat{q}^{\mu\nu}u_\nu = (1+\beta N)u^\mu$ and the magnitude of u_μ in $\hat{q}_{\mu\nu}$-metric is

$$\hat{N}_{(q)} \equiv \hat{u}^\mu u_\mu = (1+\beta N)N,$$

which is imposed by Eq. (27) to be constant different from zero. For convenience, we set $\hat{N}_{(q)} = 1$.

A power law expansion in terms of $\epsilon \sim r_H/r$ of metric (27), which corresponds to the weak field limit, gives the following expressions

$$\begin{aligned} \hat{q}_{00} &\approx 1 - \frac{r_H}{r}\left(1 + \frac{r_H}{r}\right) + \mathcal{O}(\epsilon^3),\\ \hat{q}_{01} &\approx -\left(\frac{r_H}{r}\right)^{3/2}\left(1 + \frac{r_H}{r}\right) + \mathcal{O}(\epsilon^{7/2}),\\ \hat{q}_{11} &\approx -1 - \left(\frac{r_H}{r}\right)^2\left(1 + \frac{r_H}{r}\right) + \mathcal{O}(\epsilon^4). \end{aligned} \quad (29)$$

We see that this metric does not correspond to linearized Schwarzschild solution (even in Painlevé-Gullstrand coordinates due to the power $3/2$ instead of $1/2$ in $\hat{q}_{01}$). This metric is similar to some post-Newtonian approximation if we consider a rectilinear moving source for the gravitational field. The angular components of the metric are identical to Minkowski ones.

Case II: let us consider another geometry $\hat{n}_{\mu\nu}$ given by

$$\hat{n}_{\mu\nu} = \eta_{\mu\nu} + Bu_\mu u_\nu + \Delta a_\mu a_\nu + \Lambda a_{(\mu}u_{\nu)}. \quad (30)$$

The metric components are explicitly written as

$$\begin{aligned} \hat{n}_{00} &= 1 + B, \qquad \hat{n}_{01} = f(B - \Lambda f'),\\ \hat{n}_{11} &- 1 + f^2(B + \Delta f'^2 - 2\Lambda f'). \end{aligned} \quad (31)$$

The other spatial components are identical to Minkowski metric in spherical coordinates. The condition which led u_μ to follow a geodesic motion in this metric is imposed on its magnitude

$$\hat{N}_{(n)} = \frac{N - \Delta(a^2N + \dot{N}^2/4)}{\mathcal{Z}} \equiv \text{const}, \quad (32)$$

where $\hat{N}_{(n)} \equiv \hat{n}_{\mu\nu}u^\mu u^\nu$. We also define

$$\mathcal{Z} = \left(1 + \frac{\dot{N}\Lambda}{2}\right)^2 - a^2\Delta + NB + a^2N\Lambda^2 - B\Delta\left(a^2N + \frac{\dot{N}^2}{4}\right).$$

In this case, we can set either $\hat{N}_{(n)} = 0$ or $\hat{N}_{(n)} = 1$. Let us analyze both cases separately:

Case II.a: if $\hat{N}_{(n)} = 0$, the assumption (32) implies that

$$\Delta = \frac{N}{a^2N + \dot{N}^2/4}.$$

Once u_μ is nulllike in $\hat{n}$—metric, we expect to regain the metric component of the PPN approximation responsible to describe correctly the light propagation deflection, which is given by the $\hat{n}_{11}$ component of the metric. So, using the considerations above, we set

$$\begin{aligned} \hat{n}_{00} &\approx 1 + \frac{r}{r_H} - \frac{r_H}{r} + \mathcal{O}(\epsilon^2), \qquad \hat{n}_{01} \approx 0,\\ \hat{n}_{11} &\approx -1 - \frac{r_H}{r} + \left(\frac{r_H}{r}\right)^2 + \mathcal{O}(\epsilon^3). \end{aligned} \quad (33)$$

Note that a strange linear term appears in $\hat{n}_{00}$ due to our assumptions. If we try to avoid this term making some coordinate transformation, then the asymptotically flat regime is lost by other metric components, in such way that the trouble persists.

Case II.b: in the case of $\hat{N}_{(n)} = 1$, we can correctly reproduce the linear approximation of Schwarzschild solution. That is,

$$\begin{aligned} \hat{n}_{00} &\approx 1 - \frac{r_H}{r} + 2\left(\frac{r_H}{r}\right)^2 + \mathcal{O}(\epsilon^3), \qquad \hat{n}_{01} \approx 0,\\ \hat{n}_{11} &\approx -1 - \frac{r_H}{r} + \mathcal{O}(\epsilon^3). \end{aligned} \quad (34)$$

Note that $\hat{n}_{00}$ differs from PPN results already in second order approximation in ϵ (whose the expected value in ϵ^2 is $1/2$ instead of 2). Therefore, it will surely produce some discrepancy in high order terms of the expansion.

Remark that these calculations were basically done in order to illustrate some analogies between these geometries and those studied in analog models of gravity. For this reason, the first order approximation is enough to show their strong correlation. If someone takes this approach looking for a perfect kinematical analogy between Schwarzschild metric and some $\hat{n}_{\mu\nu}$, then the introduction of a nonlinear scalar potential Ψ in Newtonian equation of acceleration becomes necessary and, therefore, it can reproduce exactly the Schwarzschild geodesics, for instance. Notwithstanding, we will not enter into the details of this generalization because it involves a complicated question about the physical meaning of the dynamics of such nonlinear potential, whereas in this paper we want to discuss only kinematical properties of the particle trajectory.

V. CONCLUSION

In this paper, we presented an extension of the Gordon metric constructing a larger class of geometries which describes accelerated motions in Minkowski space-time as geodesics in an effective geometry $\hat{q}_{\mu\nu}$. This effective metric depends only on the background metric and on the velocity vector of the accelerated body. In particular, we analyzed accelerated paths of light inside a moving dielectric and constructed a class of geometries with peculiar properties in comparison to Gordon's approach, but with similar kinematical effects. Ultimately, we gather this new class of geometries, that we call $\hat{Q}$-metrics, with the effective geometries of nonlinear electromagnetism, producing a collection of possible metric structures of the space-time, which has several applications in the theory of analog models of gravity. We will come back to this in the future.

ACKNOWLEDGMENTS

We acknowledge J. M. Salim, J. D. Toniato, E. Goulart, V. de Lorenci and R. Klippert for their comments and the staff of ICRANet in Pescara where this work was partially done. We would like to thank FINEP, FAPERJ, CNPq and CFC/CBPF for their financial support.

APPENDIX: HADAMARD'S METHOD FOR DISCONTINUITIES

We analyze the discontinuities of the electromagnetic field according to the standard Hadamard method and obtain the dispersion relation for the wave vector k_μ. Let Σ be a surface of discontinuity of the field A_μ. The discontinuity of an arbitrary function f is given by:

$$[f(x)]_\Sigma = \lim_{\epsilon \to 0^+} (f(x+\epsilon) - f(x-\epsilon)). \tag{A1}$$

The field A_μ and its first derivative $\partial_\nu A_\mu$ are continuous across Σ, while the second derivatives present a discontinuity:

$$[A_\mu]_\Sigma = 0, \tag{A2}$$

$$[\partial_\nu A_\mu]_\Sigma = 0, \tag{A3}$$

$$[\partial_\alpha \partial_\beta A_\mu]_\Sigma = k_\alpha k_\beta \xi_\mu(x), \tag{A4}$$

where $k_\mu \equiv \partial_\mu \Sigma$ is the propagation vector and $\xi_\mu(x)$ is the amplitude of the discontinuity. Substituting these discontinuity properties in the equation of motion

$$\eta^{\alpha\nu} F_{\mu\nu;\alpha} \equiv \eta^{\alpha\nu} A_{[\mu,\nu];\alpha} = 0,$$

it follows that:

$$k_\alpha k_\beta \eta^{\alpha\beta} = 0.$$

This means that the discontinuities of the electromagnetic field propagate as null geodesics in the Minkowski metric $\eta_{\mu\nu}$.

[1] W. Gordon, Ann. Phys. (Leipzig) **72**, 421 (1923); F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and, Metric*, Progress in Mathematical Physics, Vol. 33 (Birkhäuser, Boston, 2003).
[2] M. Novello, V. A. De Lorenci, J. M. Salim, and R. Klippert, Phys. Rev. D **61**, 045001 (2000) and references therein.
[3] J. B. Pendry *et al.*, Science **312**, 1780 (2006).
[4] L. Landau and E. Lifshitz, *Electrodynamique des Milieux Continus*, edited by Mir (Moscow, 1969).
[5] J. Hadamard, *Leçons sur la Propagation des ondes et les Équations de Hydrodynamique*, edited by Hermann (Paris, 1903); V. D. Zakharov, *Gravitational Waves in Einstein's Theory* (John Wiley & Sons, Inc., New York, 1973).
[6] M. Novello and E. Goulart, Classical Quantum Gravity **28**, 145022 (2011) and references therein.
[7] M. Novello, S. E. Perez-Bergliaffa, and J. M. Salim, Phys. Rev. D **69**, 127301 (2004).
[8] M. Novello, S. E. Perez-Bergliaffa, J. M. Salim, V. A. De Lorenci, and R. Klippert, Classical Quantum Gravity **20**, 859 (2003).
[9] S. M. Hawking, Nature (London) **248**, 30 (1974).
[10] W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
[11] M. Novello, M. Visser, and G. Volovik (Eds), *Proceedings of the Workshop: Analog Models of General Relativity* (World Scientific, Singapore, 2002).
[12] C. Barceló, S. Liberati, and M. Visser, Living Rev. Relativity **14**, 3 (2011).
[13] R. Feynman *et al.*, *Feynman Lectures on Gravitation* (Westview Press, Boulder, 1995), 1 ed.
[14] D. R. Smith *et al.*, Science **305**, 788 (2004).
[15] D. Scuring, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, Science **314**, 977 (2006).
[16] C. Will, Living Rev. Relativity **9**, 3 (2006).
[17] H. Poincaré, *Science and Hypothesis*, edited by Scott (Michigan University, New York, 1905).